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FOURFOLD GEOMETRY

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BEING THE ELEMENTARY GEOMETRY OF THE FOUR-DIMENSIONAL WORLD

BY

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WITH 61 DIAGRAMS

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PREFACE

IN consequence of the revolution produced in mathematical thought by the principle of relativity, geometry falls naturally into three stages. The first stage is the geometry of Euclidian space. The second stage brings the fourth dimension of the world into consideration and assumes the existence of straight lines. The third stage continues the discussion of the four-dimensional world and recognizes that straight lines do not exist and that their place is taken by geodesics. It is the second stage that is treated in this book.

If we want to study the structure of an apple we may cut it in half, and thus obtain more knowledge. Further sections will give additional knowledge. But if we could make the apple transparent instead of cutting it up we should obtain a truer picture. The classical view of the world, which treats it as three-dimensional, gives only a section of the true world which has four dimensions. Our aim is to study the four-dimensional world without cutting it up, and thereby to obtain a picture that is as much truer than the classical view as our picture of the transparent apple is truer than the view we obtain by cutting it in half.

The chief importance of the study of the "flat" fourfold lies in the fact that it is a necessary preliminary to the study of the curved fourfold. But it has also value of its own. This limited treatment has, for example, led to important improvements in the laws of electromagnetic phenomena, and has revolutionized the laws of motion. The changes in the laws of motion do not affect relatively slow motions such as engineers deal with, but are of the first consequence for rapidly moving bodies like electrons.

The demands on the knowledge of one who wishes to read this book are not great. He must be acquainted with the Cartesian method of treating geometry and with the meaning of conjugacy with respect to a conicoid.

Lastly, a word on terminology. English, like other languages, was evolved by a race conscious only of three dimensions. In consequence, it is seriously inadequate to the description of the fourfold world. The circumlocutions necessary in the absence of new terms would be intolerable. The invention of new terms is unavoidable, but satisfactory terms cannot be invented in a moment, and we ask the indulgence of the reader for their inevitable inadequacy.

I thank my colleague, Dr. H. B. Heywood, for many valuable suggestions made during proof-reading.

D. B. M.

April, 1926

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FOURFOLD GEOMETRY

CHAPTER I

VECTOR QUANTITIES

Loci of Various Dimensions

1. AN unrestricted point in a manifold has degrees of freedom equal in number to the dimensions of the manifold. One degree of restraint defined by an equation or otherwise removes a degree of freedom, and every additional degree of restraint removes another degree of freedom. Thus in Euclidian space of three dimensions one equation picks out a two-dimensional region which we call a surface, or in special circumstances a plane; two equations pick out a curve which may in particular cases be a straight line; and three equations give one or more points.

In the fourfold an equation picks out a three-dimensional region. This region we shall call a trisurface, and in contradistinction the surface of two dimensions will be called a bisurface. Corresponding to these we shall also use the terms triplane and biplane.

Two equations in a fourfold give a locus of two dimensions, a bisurface. Three equations give a line or curve, and four equations give one or more points.

In consequence two triplanes meet in general in a biplane, and three meet in a line; a triplane and a biplane meet in a line, a triplane and a line meet in a point; two biplanes meet in a point, and a biplane and a line do not in general meet at all.

Line Vector

2. A straight line has direction and length. When considered in relation to its direction as well as its length it is called a line-vector or vector. We distinguish its two ends, calling one the head and the other the foot; the positive direction of a vector is from the foot to the head.

In the fourfold which is our world, time enters as well as space. In keeping with this fact we shall speak of the size or magnitude of a line or vector in place of the length, as the terms size and magnitude are less definitely limited to space measurement than the term length.

Area Vector

3. Two vectors **k l** with their feet at the same point *O* determine a parallelogram and a biplane, namely, the parallelogram of which they form two sides, and the biplane in which they lie. The perimeter of the parallelogram may be traversed in either of two directions. We denote it by **kl** when the circuit starts from *O* along **k**. We also denote it by a single letter **a**.

4. Letters denoting vector quantities are printed in black type to distinguish them from quantities in which magnitude alone is expressed without regard to direction or orientation. For the latter italic type is used. In manuscript work it is convenient to use underlining to denote vector quantities.

5. The parallelogram $\mathbf{a} = \mathbf{kl}$ may be thought of as generated by moving the foot of vector **k** along **l**, the size

and direction of **k** remaining unaltered, or as generated by such a parallel shift of **l** along **k**

An ambiguity lurks in the symbol which may mean "is equal to" and belong to a principal clause or may mean "which is equal to" and belong to a subordinate clause. To avoid that ambiguity we restrict the symbol to use in subordinate clauses, that is to meanings like: which is identical with, also written as, also denoted by, which is the same thing as.

6. Multiplying **k** or **l** by a number multiplies the parallelogram by the same number. Thus

$$(\kappa\mathbf{k})\mathbf{l} = \kappa\mathbf{a} \qquad \mathbf{k}(\lambda\mathbf{l}) = \lambda\mathbf{a}$$

If the multiplier is -1 we have

$$(-\mathbf{k})\mathbf{l} = -\mathbf{a} \qquad \mathbf{k}(-\mathbf{l}) = -\mathbf{a}$$

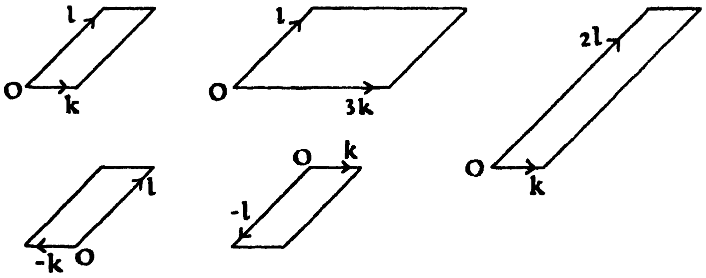

Figure for article 1,6.

Multiplication of one of the vectors by -1 reverses the direction in which the perimeter is traversed, so that if **a** is the appropriate name for the circuit traversed in one direction the appropriate name for the other direction is $-\mathbf{a}$.

7. The parallelogram has two faces or aspects and we associate each of the two directions of the circuit with one of the two aspects.

The circuit can begin at any corner of the parallelogram. Thus, by starting from different corners, taking the circuit always the same way round, using the two vectors drawn from the corner in question, and naming first the one we start along, we obtain the following equivalent descriptions

$$\mathbf{a} = \mathbf{k}\mathbf{l} = \mathbf{l}(-\mathbf{k}) = (-\mathbf{k})(-\mathbf{l}) = (-\mathbf{l})\mathbf{k}$$

and by taking the opposite circuit

$$-\mathbf{a} = \mathbf{l}\mathbf{k} = (-\mathbf{k})\mathbf{l} = (-\mathbf{l})(-\mathbf{k}) = \mathbf{k}(-\mathbf{l})$$

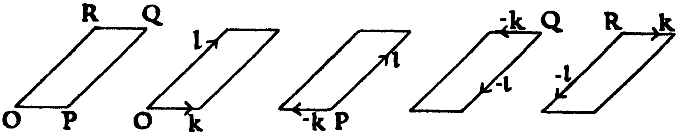

Figure for article 1,7.

8. The biplane determined by **k** **l** has a certain orientation, that is to say certain inclinations to other figures in the fourfold; the orientation of two parallel biplanes is the same.

The parallelogram $\mathbf{a} \equiv \mathbf{k}\mathbf{l}$ considered in respect of its size, orientation, and aspect * is called an area-vector. Any piece of a biplane considered as having size, orientation, and aspect is called an area-vector. In general the form of the boundary is immaterial and any area-vector can be replaced by an equivalent parallelogram.

9. We speak of making the circuit of the perimeter *OPQRO* of a parallelogram. That does not mean that we conceive a material particle travelling round. We shall see later that that is impossible because it would sometimes

---

* Aspect refers to one of two alternatives, namely, one or other of the two faces from which a biplane or a triplane can be considered to look out. Orientation refers in a wider way to the relative location of the various parts of a figure in its surroundings.

require the particle to travel backwards in time and sometimes require it to travel faster than light. By describing the circuit in this direction we only mean that we think first of vector $\overrightarrow{OP}$, then of $\overrightarrow{PQ}$, then of $\overrightarrow{QR}$, and finally of $\overrightarrow{RO}$.

10. In Euclidian space we connect (i) a particular direction of looking out of a plane with (ii) a circuit in the plane, in the way that (i) the direction of motion of a screw is connected with (ii) the direction in which the screw rotates. The analogy fails in the fourfold, both because there is a whole biplaneful of directions normal to a given biplane, and also because (as we shall see) one vector cannot always be rotated to lie along another.

Volume Vector

11. Three vectors **k l m** with their feet at the same point O determine the parallelepiped of which they form three edges, and determine the triplane in which they lie. The triplane has two faces which look out on the two pieces into which it divides the fourfold. We associate one of these faces or aspects with the name **klm** for the parallelepiped; we also denote **klm** by a single letter **v**.

12. To multiply one of the vectors by a number multiplies the parallelepiped by the same number, that is

$$\rho\mathbf{v} = (\rho\mathbf{k})\mathbf{lm} = \mathbf{k}(\rho\mathbf{l})\mathbf{m} = \mathbf{kl}(\rho\mathbf{m})$$

and when the multiplier is -1 we have

$$-\mathbf{v} = (-\mathbf{k})\mathbf{lm} = \mathbf{k}(-\mathbf{l})\mathbf{m} = \mathbf{kl}(-\mathbf{m})$$

13. The parallelepiped **klm** may be generated by moving the area vector $\mathbf{a} \equiv \mathbf{kl}$ parallel to itself, one corner of it

traversing the vector **m**; or by moving vector **m** parallel to itself, its foot ranging all over the area vector **a**.

14. Think of the parallelepiped $\mathbf{v} = \mathbf{klm}$ as generated by the parallel shift of **a** along **m**. The multiplication of **a** by a number multiplies **v** by the same number. When the number is -1 this gives

$$-\mathbf{v} = (-\mathbf{a})\mathbf{m} = \mathbf{lkm}$$

so that the interchange of vectors **k l** changes the sign of **v**.

We can look on **v** as generated otherwise, say by the shift of **lm** along **k**, and in that way see that the interchange of **l** and **m** changes the sign of **v**. Thus the interchange of any two vectors changes the sign of **v**, that is

$$\begin{aligned} \mathbf{v} &= \mathbf{klm} = \mathbf{lmk} = \mathbf{mkl} = \mathbf{am} = \mathbf{ma} \\ -\mathbf{v} &= \mathbf{mlk} = \mathbf{lkm} = \mathbf{kml} \end{aligned}$$

15. The triplane has an orientation in the fourfold, and the parallelepiped considered in respect of size, orientation, and aspect is called a volume-vector. And any piece of the same triplane of the same size and considered to have the same aspect is called an equal volume-vector.

The Tetraped

16. Along with three vector lines **l m n** forming a parallelepiped we take a fourth **p**, and we move the parallelepiped parallel to itself, one corner traversing **p** from foot to head. The figure generated is the fourfold analogue of the parallelepiped in three dimensions.

The word parallelepiped is itself unwieldy, and if we added to it in order to name the analogue and to distinguish between the original and the analogue the new words would be of unconscionable length. We shall therefore telescope them to triped and tetraped, in keeping with which a parallelogram would be a dyped.

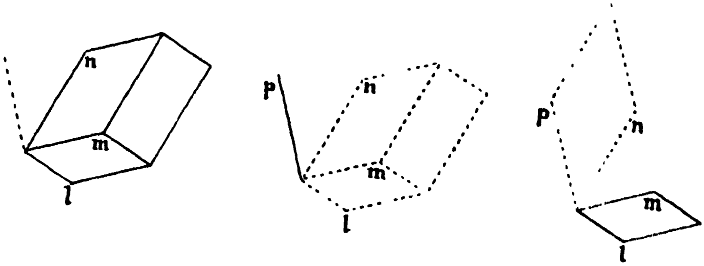

Figure for article 1,16.

17. The tetraped, having as many dimensions as the fourfold, will have no orientation, any more than a triped has in a threefold or a parallelogram in a twofold. It is a scalar quantity, and an italic letter *f*, and not a black letter, is appropriate to denote it. We shall call it a supervolume to distinguish it from a three-dimensional volume. We write

$$f = \mathbf{lmnp}$$

18. The tetraped can also be generated by moving line vector **p** parallel to itself, its foot ranging all through the volume **lmn**, or by moving the area vector **lm** parallel to itself, one corner ranging all over the area vector **np**.

19. Multiplication of one of the four vectors by a number multiplies the tetraped by the same number, so that when the number is -1 we have

$$-f = (-\mathbf{l})\mathbf{mnp} = \mathbf{l}(-\mathbf{m})\mathbf{np} = \mathbf{lm}(-\mathbf{n})\mathbf{p} = \mathbf{lmn}(-\mathbf{p})$$

Again to multiply $\mathbf{a} = \mathbf{lm}$ or $\mathbf{v} = \mathbf{lmn}$ by -1 also multiplies f by -1 so that

$$-f = (-\mathbf{a})\mathbf{np} = \mathbf{mlnp}$$
$$-f = (-\mathbf{v})\mathbf{p} = \mathbf{lnmp} = \text{etc.}$$

Thus we see that any interchange of two line-vectors changes the sign of *f*.

20. The tetraped is bounded by eight tripeds. One is **lmn** and another this triped in the position to which the shift along **p** brings it. If we generate the tetraped by shifting **mlp** along **n**, we see that another pair are **mlp** in its initial and final positions. Similarly by shifting **lnp** along **m**, and **mpn** along **l**, we obtain the remaining two pairs of bounding tripeds.

The last six of these tripeds we can also look on as the figures generated by the parallel shift of the six faces of **lmn** along **p**.

21. If a pair of boundary tripeds meet, the interface is a parallelogram, for example **lmn** and **mlp** meet in the parallelogram **lm**. And **lmn** or any other of the tripeds meets six others, one on each of its faces. Each of the eight tripeds has six faces, and each interface counts twice, so that there are 24 interfaces.

Three tripeds meet in a line if they meet at all. Each triped has 12 edges and each edge is common to three tripeds, so that the tetraped has 32 edges. Four tripeds meet in a point and each triped has 8 corners, so that the tetraped has 16 corners.

22. We could also count the interfaces as the six faces of **lmn**, the six into which they shift, and the twelve generated by the shift of the twelve edges of **lmn**, in all 24. And the edges we could count as the 12 edges of **lmn**, the 12 into which they shift, and the 8 generated by the shift of the 8 corners of **lmn**, a total of 32.

CHAPTER 2

THE REFERENCE LATTICE. PARALLEL PROJECTION

The Reference Lattice in Euclidian Space

1. IN the Cartesian treatment of Euclidian geometry we use three axes of coordinates and locate a point P by three coordinates. We shall denote these coordinates by x^1 x^2 x^3 where the numbers do not denote powers but are merely affixes to distinguish the three numbers. Powers will rarely be written by means of indices, and when any ambiguity is possible the quantity to be raised to a power will be bracketed.

To say that x^1 x^2 x^3 are the coordinates of P means in the classical usage that we may go from the origin O to P by travelling a distance x^1 along the first coordinate axis, then x^2 parallel to the second, and then x^3 parallel to the third. We shall depart from that usage by marking off standard lengths e_1 e_2 e_3 along the three axes, and say that if x^1 x^2 x^3 are the coordinates of P we arrive at P by travelling distances of x^1e_1 x^2e_2 x^3e_3 parallel to the three axes.

2. We also use $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ in black type to denote vectors of lengths e_1 e_2 e_3 along the three axes. Then $x^1\mathbf{e}_1$ is also a vector and states both the direction and distance that we

are to go parallel to the first axis; and so for $x^2\mathbf{e}_2$ $x^3\mathbf{e}_3$. A vector sum such as

$$x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$$

is an instruction to go certain distances along the three vectors in succession, that is it is a description of our manner of travelling from O to P. We use $\overrightarrow{OP}$ to denote the vector line OP and express this fact by writing

$$\overrightarrow{OP} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$$

We also denote the vector line OP by $\mathbf{x}$ in heavy type, so that

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$$

The vectors $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ we call the prime vectors of the reference system. The quantities x^1 x^2 x^3, which are the coordinates of the point P, will also be called the components of the vector $\mathbf{x}$.

3. There is a convention that enables us to write certain expressions with great conciseness. It is that when, as in

$$x^a\mathbf{e}_a$$

a literal affix appears twice, that is an instruction to give a the three values 1 2 3 and add, so that $x^a\mathbf{e}_a$ means $x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$. In fourfold geometry the repetition of an affix instructs us to sum for the four values 1 2 3 4.

4. All along the first coordinate axis we mark off lengths of e_1 in both positive and negative directions, and through each mark we draw a plane parallel to $\mathbf{e}_2$ and $\mathbf{e}_3$. The plane containing $\mathbf{e}_2$ and $\mathbf{e}_3$ we will call the plane $\mathbf{e}_{23}$. In the same way we divide up the second and third axes and draw planes parallel to $\mathbf{e}_{31}$ and $\mathbf{e}_{12}$.

These planes divide space into cells. Each cell is a parallelepiped; each face of a cell is the parallelogram cut from one plane by four others; each edge of a cell has the length and direction of a prime vector, and is the portion of the line of intersection of two planes cut off by two other planes; each corner is the intersection of three planes.

The totality of the edges of the cells form a network or lattice which we shall call the reference lattice.

The Reference Lattice in the Fourfold

5. In the fourfold we start from four prime vectors $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$. We produce each in both directions and mark off equal lengths as in the threefold. Through each mark on the prolongation of $\mathbf{e}_1$ we draw a triplane parallel to $\mathbf{e}_{234}$ that is parallel to the triplane containing $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$. In the same way we draw the other three series of triplanes.

These four series of triplanes divide up the fourfold into supercells which are tetrapeds. The eight tripeds that bound a supercell are parallel in pairs to $\mathbf{e}_{234}$ $\mathbf{e}_{143}$ $\mathbf{e}_{412}$ $\mathbf{e}_{321}$. The 24 parallelograms in which the tripeds meet are parallel in fours to the six biplanes $\mathbf{e}_{12}$ $\mathbf{e}_{13}$ $\mathbf{e}_{14}$ $\mathbf{e}_{23}$ $\mathbf{e}_{34}$ $\mathbf{e}_{41}$. And the 32 edges of a supercell are parallel eight by eight to the four prime vectors

6. **Exercise.**—How many supercells have a corner at the origin? How many have the prime vector $\mathbf{e}_1$ for an edge? How many have $\mathbf{e}_{12}$ as one of their parallelograms? How many share the triped $\mathbf{e}_{123}$?

7. **Exercise.**—Show how to draw triplanes, parallel to the prime triplanes, through the centres of the 16 tetrapeds which meet at the origin, so as to enclose the origin. Discuss how many corners, how many edges, how many parallelogram faces, and how many triped faces of supercells lie in the enclosure.

Hence state how many of each of these constituents the lattice has per supercell.

8. Being given the coordinates x^1 x^2 x^3 x^4 of the point P, or what is the same thing being given the vector

$$x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + x^4\mathbf{e}_4$$

we arrive at the point P by making the appropriate traverses in succession parallel to the four prime vectors.

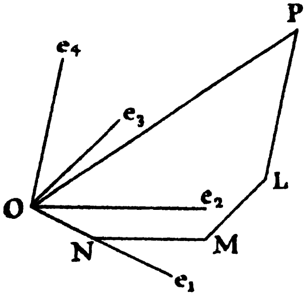

Figure for article 2,8.

If we are given the point P but not its coordinates, or if we are given the vector $\mathbf{x}$ or $\overrightarrow{OP}$ but not its components, to find these coordinates or components we draw through P a line parallel to $\mathbf{e}_4$. A line and a triplane meet in a point so that this line meets $\mathbf{e}_{321}$ in a point L. We are now concerned with the triplane $\mathbf{e}_{321}$ which is a threefold within the fourfold. In this threefold we draw LM parallel to $\mathbf{e}_3$ to meet the biplane $\mathbf{e}_{12}$ in M. Then in the twofold $\mathbf{e}_{12}$ we draw MN parallel to $\mathbf{e}_2$ to meet $\mathbf{e}_1$ in N.

We have now a route from O to P each portion of which is parallel to a prime vector and x^1 x^2 x^3 x^4 are given by

$$x^1\mathbf{e}_1 = \overrightarrow{ON} \quad x^2\mathbf{e}_2 = \overrightarrow{NM} \quad x^3\mathbf{e}_3 = \overrightarrow{ML} \quad x^4\mathbf{e}_4 = \overrightarrow{LP}.$$

9. In the fourfold the repetition of a literal affix, as in $x^\theta\mathbf{e}_\theta$, is an instruction to give the affix the four values 1 2 3 4 and add; so that we write

$$\overrightarrow{OP} = \mathbf{x} = x^\theta\mathbf{e}_\theta$$

10. When we specify the coordinates of a point we are specifying a place and a time, we are saying that something

happened at a certain place at a certain time, that is we are specifying an event or point-event. Point-event is the appropriate name for the thing located by four coordinates in the fourfold.

We speak of the distance between two places and of the interval between two moments of time. We want a corresponding term when the distance or interval involves both space and time. We shall use the term separation between two point-events to mean the vector that joins the two. We shall also speak of the size or the magnitude of a separation.

11. If we endeavour to build up the supercells of our lattice till they fill the fourfold we meet with difficulties. The fourfold is warped or curved, it does not contain straight lines and geodesics have to do duty instead of them, and the direction of the edge of a supercell depends upon the route by which we arrive there in the building up.

We shall confine ourselves to a region of the fourfold within which the coordinates are of moderate magnitude. When we do that, and also choose an appropriate reference lattice, the geodesics do not differ perceptibly from straight lines and can be treated as straight lines. This region in which the geodesics are straight we shall call the "local region" of the lattice.

The appropriate kind of reference lattice is one for which, in the local region, that is to say in the neighbourhood of the origin, the world line of a ray of light is straight or in other words, one for which light travels with constant velocity.

12. The meaning of parallelism is a question fundamental to an attempt to extend our reference lattice indefinitely. The answer to the question is a matter for the higher

geometry of the fourfold. We confine ourselves in this book to the elementary geometry of the fourfold, that is to say to the geometry of the local region, in which the geodesics are straight and parallelism has the ordinary Euclidian meaning.

The Surface Plate

13. One test of the absence of curvature for a small portion of the fourfold is the possibility of making true planes in the Euclidian sense. A sheet of steel with a true plane for one surface is called a surface plate. To ensure planeness three surface plates are made at the same time, they are ground against one another very carefully until every pair of them fit with the greatest nicety. The grinding of two plates against one another will not ensure planeness, as one might be concave and the other convex, but a third plate cannot fit both of them unless all three are plane.

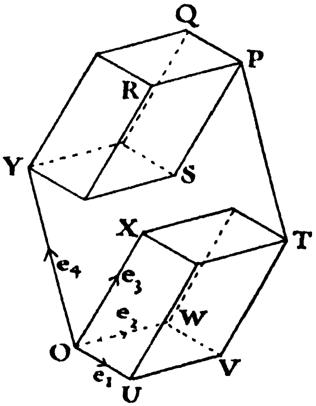

Figure for article 2,14.

Parallel Projection of a Point-Event

14. The projection of a point or point-event parallel to a line on to a triplane is closely analogous to the projection in Euclidian space of a point parallel to a line on to a plane. Thus if from P we draw PT parallel to $\mathbf{e}_4$ to meet $\mathbf{e}_{123}$ in T, T is the projection of P, and TP the projecting line or projector is $x^4\mathbf{e}_4$ where

$$\overrightarrow{OP} = \mathbf{x} = x^a\mathbf{e}_a = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + x^4\mathbf{e}_4$$

We complete the broken line $OUVTP$ as before so that

$$\overrightarrow{OU} = x^1\mathbf{e}_1 \quad \overrightarrow{UV} = x^2\mathbf{e}_2 \quad \overrightarrow{VT} = x^3\mathbf{e}_3$$

Then we complete the triped of which OU UV VT are edges, and shift the triped parallel to itself along TP to the position shown in the figure. Q is the projection of P parallel to $\mathbf{e}_1$ on the lattice triplane $\mathbf{e}_{234}$ since PQ is parallel to $\mathbf{e}_1$ and Q lies in $\mathbf{e}_{234}$. Similarly R is the projection parallel to $\mathbf{e}_2$ on the triplane $\mathbf{e}_{134}$, and S the projection parallel to $\mathbf{e}_3$ on $\mathbf{e}_{124}$.

15. The projection of P parallel to the triplane $\mathbf{e}_{234}$ upon the line $\mathbf{e}_1$ is also analogous to a projection in Euclidian space. We draw through P the triplane with the appropriate orientation and its intersection with $\mathbf{e}_1$ is the required projection. Since UV VT TP are parallel to $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ the triplane determined by these three lines is parallel to $\mathbf{e}_{234}$. It is thus the triplane of projection and U is the projection of P.

In fact the shift of the triped $OUVT$ generated a tetraped of which four bounding triplanes pass through O and the other four through P. OU and QP are a pair of parallel edges which join $\mathbf{e}_{234}$ and the parallel triplane through P, and are thus the projection of OP parallel to $\mathbf{e}_{234}$ upon $\mathbf{e}_1$ and the projector of OP parallel to $\mathbf{e}_1$ upon $\mathbf{e}_{234}$. And so for the other pairs OW RP; OX SP; OY TP.

16. We observe that the projecting lines that project different points parallel to $\mathbf{e}_{234}$ upon $\mathbf{e}_1$ are not parallel to one another.

17. How are we to project the point-event P upon a biplane, say upon $\mathbf{e}_{12}$? We cannot project parallel to a line because a line and a biplane do not meet. To project

parallel to another biplane is appropriate since two biplanes meet in a point.

Let us project P parallel to $\mathbf{e}_{34}$ upon $\mathbf{e}_{12}$. The projection is V, for the parallelogram $PTVS$ is parallel to $\mathbf{e}_{34}$ and V is the point it has in common with $\mathbf{e}_{12}$.

18. We notice that when a number of events are projected parallel to $\mathbf{e}_{34}$ upon $\mathbf{e}_{12}$, the projecting lines all lie parallel to the biplane $\mathbf{e}_{34}$ but are not in general parallel to one another.

19. **Exercise.**—In the figure of article 2,14 locate the projections of the vector $OP \equiv \mathbf{x} \equiv x^a\mathbf{e}_a$ upon

$$\mathbf{e}_{12} \quad \mathbf{e}_{13} \quad \mathbf{e}_{14} \quad \mathbf{e}_{23} \quad \mathbf{e}_{34} \quad \mathbf{e}_{42}$$

the projections being made parallel to

$$\mathbf{e}_{34} \quad \mathbf{e}_{42} \quad \mathbf{e}_{23} \quad \mathbf{e}_{14} \quad \mathbf{e}_{12} \quad \mathbf{e}_{13}$$

respectively. Also give vectorial expressions for the projections.

CHAPTER 3

THE FUNDAMENTAL QUADRATIC FUNCTION

1. AN observer O furnished with a rectangular set of three axes, a measuring rod, and a watch locates a point P as having coordinates x y z and locates an event E as happening at time t. He finds the distance between two points P P^1 to be

$$\sqrt{(x^1 - x)^2 + (y^1 - y)^2 + (z^1 - z)^2}$$

and the time-lapse between two events E E^1 to be $t^1 - t$.

Let observers K L . . . occupy fixed positions with respect to O's reference frame, and measure the distance PP^1 and the time-lapse EE^1. In both classical and relativity geometry their measures of these quantities will be the same as O's, and we say that these quantities are invariant for all such observers.

Let observers Q R . . . move relative to O with constant velocities. Classical geometry says that for them also the distance PP^1 and the time-lapse EE^1 are invariant, but relativity geometry denies the invariance of both quantities and sets up in their place the invariant

$$\sqrt{(x^1 - x)^2 + (y^1 - y)^2 + (z^1 - z)^2 - (t^1 - t)^2}$$

in which the units of space and time are so related as to make the velocity of light unity. Relativity geometry says this quantity is invariant provided that we choose for O a set of axes relative to which light travels in straight lines, and provided that the magnitude of our coordinates is so moderate as not to be affected by the warping of the world.

It is a hypothesis of the relativity theory that the speed of light has the same constant value for all reference systems in which light travels in straight lines.

In relativity geometry we do not discuss the position of point P regardless of time, or the time of an event E regardless of position; what we discuss is a point-event P happening at the place $x\ y\ z$ at the time t, or as we may state it a point-event with the four coordinates $x\ y\ z\ t$.

2. To the reference frame or reference lattice carried by O we add a fourth axis, which we look on as perpendicular to the original three axes, and along this we measure t. In the four-dimensional manifold constituted by the four axes of reference we draw the vector $\overrightarrow{OP}$ which has the four components $x\ y\ z\ t$.

We square the invariant quantity of article 1 and change the sign, and take P^1 to lie at the origin, and thus have as an invariant related to OP

$$t^2 - x^2 - y^2 - z^2$$

This quantity which defines the nature of the fourfold world is called the Fundamental Quadratic Function.

The invariance of this quadratic function of the coordinates is to begin with only a hypothesis. It and other hypotheses of the theory of relativity are raised to the status of established fact by the marvellous accuracy with which the theory fits the data of physical science.

3. We observe that when $x\ y\ z$ are zero the fundamental quadratic function is the square of a time, and that when t

is zero it is minus the square of a length. When the function is positive we say that vector $\overrightarrow{OP}$ is timelike, and that when the function is negative $\overrightarrow{OP}$ is spacelike. We define $\sqrt{t^2 - x^2 - y^2 - z^2}$ to be the magnitude or measure or size of a timelike vector and $\sqrt{x^2 + y^2 + z^2 - t^2}$ to be that of a spacelike vector; and in both cases we denote it by σ_{OP}. We also define $t^2 - x^2 - y^2 - z^2$ to be the size-number of OP and denote it by ν_{OP}. Thus for a timelike vector we have

$$\nu_{OP} = \sigma^2_{OP} = t^2 - x^2 - y^2 - z^2$$

and for a spacelike vector

$$\nu_{OP} = -\sigma^2_{OP} = t^2 - x^2 - y^2 - z^2$$

4. Let us consider the locus of P for which ν_{OP} has a constant value, and first we will consider the locus for which ν_{OP} is zero. The homogeneity of the expression shows that if $x \; y \; z \; t$ is on the locus so also is $kx \; ky \; kz \; kt$. That is, every point upon the line joining it to the origin is also on the locus, which is thus of the nature of a cone.

The coordinates of every point or point-event on a straight line may be expressed in the form

$$x = a + l\rho \qquad y = b + m\rho \qquad z = c + n\rho \qquad t = d + p\rho$$

in terms of the variable parameter ρ. The substitution of these values in $\nu_{OP} = 0$ gives a quadratic equation for ρ so that the locus has two intersections with a straight line.

The locus differs however from the cone of Euclidian space by having three dimensions. We shall call it a tricone and in contradistinction call a cone of two dimensions a bicone.

5. When OP lies near the t-axis ν_{OP} is positive. It may

fall away from the t-axis in such a way that x increases while y and z remain zero. In that case ν_{OP} remains positive until $x = t$ when it is zero, and beyond that it becomes negative. Similarly the size-number falls to zero when y has increased to t, x and z remaining zero, and also when z increases to t, x and y remaining zero. And generally as OP falls away in any direction from the t-axis, it reaches a position where $x^2 + y^2 + z^2 = t^2$ and the size-number is zero.

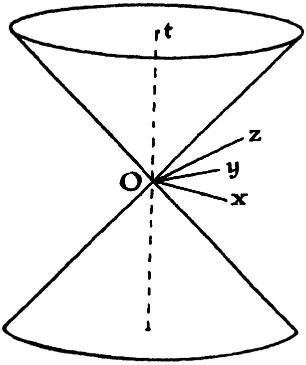

Figure for article 3,5.

Thus one half of the complete tricone surrounds the positive portion of the t-axis, while the other half surrounds the negative portion. In other words, one horn of the tricone surrounds the axis of future time and the other horn the axis of past time.

6. Consider now the locus of P for which

$$\nu_{OP} \equiv t^2 - x^2 - y^2 - z^2 = 1$$

The point-event whose coordinates are 0 0 0 1 is on the locus, and as the present locus cannot cut the tricone $\nu_{OP} = 0$, all of the locus that is continuous with this point-event must lie inside the future-time horn of the tricone. The past-time horn of the tricone contains another portion of the locus, and the whole locus is symmetrical about the origin; so that the locus consists of two sheets. It is of the second degree and may be called a trihyperboloid of two sheets. Each sheet is cuplike with the base of the cup facing the origin, as are the two sheets of the hyperboloid of Euclidian space, which we now call a bihyperboloid.

7. The locus for which

$$\nu_{OP} \equiv t^2 - x^2 - y^2 - z^2 = -1$$

contains the three point-events whose coordinates are 1 0 0 0, 0 1 0 0, 0 0 1 0. Also a portion of the locus is given by $t = 0 \quad x^2 + y^2 + z^2 = 1$, that is to say by a unit sphere which lies in the triplane $t = 0$.

If a point A whose coordinates are $a \quad b \quad c \quad d$ is such that ν_{OA} is negative (say $= -k$), there is a point of the locus on the line OA; we have only to put

$$x = \lambda a \qquad y = \lambda b \qquad z = \lambda c \qquad t = \lambda d$$

and determine λ from

$$-1 = \lambda^2(d^2 - a^2 - b^2 - c^2) = -k\lambda^2$$

If we begin with a vector of positive size-number and (keeping its foot at the origin) move it across the tricone $\nu_{OP} = 0$ its size-number becomes negative and will remain negative until the vector again meets the tricone. That is to say a vector of constant negative size-number can travel all over the region exterior to the tricone.

Thus the locus $\nu_{PO} = -1$ is a trihyperboloid of one sheet. It resembles an hourglass as nearly as a three-dimensional locus can resemble a two-dimensional one.

8. Any locus

$$\nu_{OP} \equiv t^2 - x^2 - y^2 - z^2 = k$$

is similar to the locus of article 6 or the locus of article 7 according as k is positive or negative. We shall call this locus the contour of size-number k, and the whole system of loci we shall call the contour system of the fourfold.

9. We now change the notation somewhat. We lay off vectors $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$, all of unit size, along the four axes of coordinates, and we write x^1 x^2 x^3 x^4 instead of x y z t.

The coordinates of the head of $\mathbf{e}_1$ being 1 0 0 0, its size number is $\nu_{\mathbf{e}_1} = x^4x^4 - x^1x^1 - x^2x^2 - x^3x^3 = -1$ and the size-numbers of the other prime vectors are

$$\nu_{\mathbf{e}_2} = -1 \qquad \nu_{\mathbf{e}_3} = -1 \qquad \nu_{\mathbf{e}_4} = +1$$

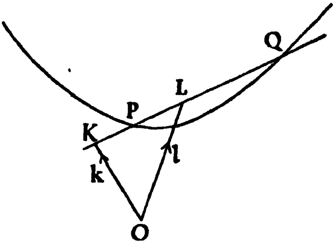

Figure for article 3,9.

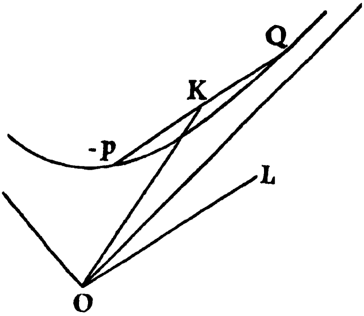

Figure for article 3,10.

Two vectors $\mathbf{k}$ $\mathbf{l}$ with their feet at the origin are said to be conjugate if they are conjugate with respect to the conics in which the biplane $\mathbf{kl}$ cuts the contour system. We use K L to denote the heads of the vectors $\mathbf{k}$ $\mathbf{l}$, and P Q to denote the points in which KL cuts the contour whose equation is $$\nu_{\mathbf{x}} = h$$
If we express the vector $\overrightarrow{OP}$ in the form $\kappa\mathbf{k} + \lambda\mathbf{l}$, the equation $\nu_{OP} = h$ becomes

$$\kappa^2\nu_{\mathbf{k}} + \lambda^2\nu_{\mathbf{l}} + 2\kappa\lambda(-k^1l^1 - k^2l^2 - k^3l^3 + k^4l^4) = h$$

The condition that $KLPQ$ forms a harmonic range, which is also the condition of conjugacy of $\mathbf{k}$ $\mathbf{l}$, is that the coefficient of $\kappa\lambda$ vanishes. Thus **k** **l** are conjugate if*

$$-k^1l^1 - k^2l^2 - k^3l^3 + k^4l^4 = 0$$

10. The biplane **kl** cuts the contour of size-number h in a conic. In the case in which the chord PQ is parallel to **l** the fact that $KLPQ$ is harmonic shows that K bisects PQ. Hence **l** is parallel to the tangent at the intersection of **k** with the conic.

When the conic is a hyperbola its asymptotes are generators of the tricone, and each asymptote is self-conjugate. By thinking of **k** **l** as coming to coincidence with one another and with an asymptote we see that a timelike and a spacelike vector, conjugate to one another, can come to coincidence on the tricone; the self-conjugate vector in which they unite being no longer either timelike or spacelike but belonging to the tricone which separates timelike from spacelike vectors.

11. The locus of vectors **x** at the origin conjugate to **k** is given by $-k^1x^1 - k^2x^2 - k^3x^3 + k^4x^4 = 0$ and is a triplane.

When a vector is conjugate to every direction in a triplane the vector and the triplane are said to be conjugate to one another.

12. We take two vectors **k** **l** at the origin and the two triplanes conjugate to them. The triplanes meet in a biplane B every line of which is conjugate to **k** and to **l**. Any vector **m** in the biplane **kl** can be expressed in the form $\kappa\mathbf{k} + \lambda\mathbf{l}$, from which we see that any vector conjugate to **k** and to **l** is conjugate to every line in the biplane **kl**. Hence every line in the biplane B is conjugate to every line in the biplane **kl**.

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* The condition is independent of h so that conjugacy with respect to one contour involves conjugacy with respect to all.

Two biplanes are said to be conjugate to one another when every direction in each is conjugate to every direction in the other.

13. Let PQ be a chord of the contour trisurface $\nu_x = h$ which is parallel to vector $\mathbf{l}$ and passes through K the head of vector $\mathbf{k}$. In this case the condition that $KLPQ$ forms a harmonic range becomes $PK = KQ$, and we see that the vector $\mathbf{k}$ drawn from the origin to bisect a chord PQ of a contour trisurface is conjugate to the chord.

14. Let a biplane parallel to the biplane determined by $\mathbf{l}$ $\mathbf{m}$ pass through the head K of the vector $\mathbf{k}$. Let KPQ drawn parallel to $\lambda\mathbf{l} + \mu\mathbf{m}$ meet the contour $\nu_x = h$ in P and Q. K is the middle point of PQ if $\mathbf{k}$ and $\lambda\mathbf{l} + \mu\mathbf{m}$ are conjugate. Hence K bisects every chord in the biplane if $\mathbf{k}$ is conjugate to $\mathbf{l}$ and to $\mathbf{m}$. In such a case we say that $\mathbf{k}$ is conjugate to the biplane. And we state the result: the vector from the origin to the centre of the conic of intersection of a biplane with a contour is conjugate to the biplane

15. Let a triplane parallel to the triplane $\mathbf{lmn}$, $\mathbf{l}$ $\mathbf{m}$ $\mathbf{n}$ being vectors at the origin, pass through the head K of vector $\mathbf{k}$. The vector $\lambda\mathbf{l} + \mu\mathbf{m} + \nu\mathbf{n}$ lies in the triplane $\mathbf{lmn}$. Parallel to this vector we draw KPQ to meet contour $\nu_x = h$ in P Q. If $\mathbf{k}$ and $\lambda\mathbf{l} + \mu\mathbf{m} + \nu\mathbf{n}$ are conjugate K bisects PQ. Hence if $\mathbf{k}$ is conjugate to $\mathbf{l}$ $\mathbf{m}$ $\mathbf{n}$, K bisects every chord parallel to $\mathbf{lmn}$; and the vector OK drawn from the origin to the centre of the biconicoid * in which a triplane cuts a contour is conjugate to that triplane.

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* Classical words for two-dimensional loci have to do duty for three-dimensional loci also. To this end we use the prefixes bi- and tri-. Thus a bisurface is any two-dimensional locus and trisurface any three-dimensional locus; a biconicoid is the conicoid of classical terminology, and a triconicoid is a three-dimensional locus that intersects a straight line in two point-events.

16. We take two vectors **l** **m** conjugate to one another. The two triplanes conjugate to **l** and **m** meet in a biplane every vector of which is conjugate to **l** and **m**. We take one of these and call it **n**. The three triplanes conjugate to **l** **m** **n** meet in a line which is conjugate to the three vectors. We use **p** to denote a vector along this line, and in **l** **m** **n** **p** we have a system of four mutually conjugate vectors.

17. We shall show that, self-conjugate vectors being excluded, a set of four mutually conjugate vectors must consist of one timelike and three spacelike vectors.

Let us first suppose that of the four **k** **l** **m** **n** the vector **k** is timelike, that is it lies inside the tricone. Let *K* lie inside the future-time horn. Since *K* bisects every chord *PQ* drawn through it and parallel to **lmn**, the ends *P* *Q* of such a chord both lie on the future-time horn. Hence if the triplane through *K* and parallel to **lmn** moves parallel to itself to coincidence with **lmn**, the ends *P* *Q* come to coincidence at the origin. Thus **lmn** meets the tricone only at the origin, that is to say it is spacelike.*

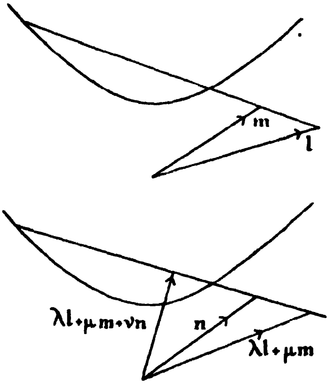

Figure for article 3,18.

18. Next let us suppose that **l** **m** **n** are spacelike

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* A biplane or triplane is called spacelike if every vector in it is spacelike. If it contains both spacelike and timelike vectors it is said to be of mixed character.

Consider the possibility that the biplane **lm** is of mixed character; in that case we should have the two vectors **l** **m** lying outside the hyperbola in which **lm** cuts the contour and at the same time conjugate with respect to it, which is impossible. Therefore biplane **lm** is spacelike.

Now consider the possibility that the triplane **lmn** contains a timelike vector, say $\lambda\mathbf{l} + \mu\mathbf{m} + \nu\mathbf{n}$. This vector lies in the same biplane with $\lambda\mathbf{l} + \mu\mathbf{m}$ and **n** which are both spacelike; and we arrive at the same contradiction as before, two vectors—$\lambda\mathbf{l} + \mu\mathbf{m}$ and **n**—in the biplane of a hyperbola, conjugate to one another, and neither meeting the hyperbola. Therefore the triplane **lmn** is spacelike.

19. Now consider the vector **k** conjugate to the triplane **lmn**. If it lay in **lmn** it would be self-conjugate and we have excluded this case. It therefore sticks out from the triplane, on the same side as one of the horns of the tricone. Say the future-time horn. The triplane *T* drawn through *K* parallel to **lmn** cuts the future-time horn only. Hence every chord of the biconicoid in which *T* cuts the tricone lies inside the tricone. *OK* being conjugate to the triplane **lmn**, *K* is the middle point of every chord of the biconicoid that passes through it, and is therefore inside the tricone Thus **k** is timelike.

We have thus shown that, apart from self-conjugate vectors, one of any set of four conjugate vectors is timelike and the other three spacelike.

20. We take a set of four conjugate vectors $\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$ $\mathbf{c}_4$, the first three being spacelike and the fourth timelike, all of unit size. We assume them to be given in terms of another set of conjugate unit vectors $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ by

$$\mathbf{c}_1 = a_1^{01}\mathbf{e}_1 + a_1^{02}\mathbf{e}_2 + a_1^{03}\mathbf{e}_3 + a_1^{04}\mathbf{e}_4 \quad . \qquad . \quad (1)$$

and three similar equations. We can write these equations concisely as

$$\mathbf{c}_1 = a_1^{\circ\rho}\mathbf{e}_\rho \quad \mathbf{c}_2 = a_2^{\circ\rho}\mathbf{e}_\rho \quad \mathbf{c}_3 = a_3^{\circ\rho}\mathbf{e}_\rho \quad \mathbf{c}_4 = a_4^{\circ\rho}\mathbf{e}_\rho \,. \qquad (2)$$

or still more concisely as

$$\mathbf{c}_\sigma = a_\sigma^{\circ\rho}\mathbf{e}_\rho \quad \sigma = 1\ 2\ 3\ 4 \,. \qquad . \qquad . \quad (3)$$

where the repetition of ρ is an instruction to give it the values 1 2 3 4 and add.

Certain relations hold among the $a_\rho^{\circ\sigma}$. Thus we have from the size-numbers

$$-1 = \nu_{\mathbf{c}_1} = (a_1^{\circ 4})^2 - (a_1^{\circ 1})^2 - (a_1^{\circ 2})^2 - (a_1^{\circ 3})^2 \quad . \quad (4)$$

and 3 other equations. There are 6 conditions of conjugacy of the vectors two and two. They may be written

$$a_\rho^{\circ 4}a_\sigma^{\circ 4} - a_\rho^{\circ 1}a_\sigma^{\circ 1} - a_\rho^{\circ 2}a_\sigma^{\circ 2} - a_\rho^{\circ 3}a_\sigma^{\circ 3} = 0 \,. \qquad . \quad (5)$$

where ρ σ have the values 1 2 3 4 and are unequal.

We can combine all ten by saying that the expression in equation 5 is to be

$$\left.\begin{array}{l} \,0 \text{ when } \rho \text{ and } \sigma \text{ are unequal} \\ +1 \text{ when } \rho = \sigma = 4 \\ -1 \text{ when } \rho = \sigma = 1\ \ 2\ \ 3 \end{array}\right\} \quad . \qquad . \quad (6)$$

21. We may use $\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$ $\mathbf{c}_4$ as a reference lattice. Suppose that a vector $\overrightarrow{OP}$ has components x^1 x^2 x^3 x^4 in the **e**-system and components u^1 u^2 u^3 u^4 in the **c**-system. We have then

$$\overrightarrow{OP} = \mathbf{u} = u^\alpha\mathbf{c}_\alpha$$

and by substituting the values of $\mathbf{c}_\alpha$

$$= u^\alpha a_\alpha^{\circ\rho}\mathbf{e}_\rho$$

Since $\overrightarrow{OP}$ is also equal to $x^\rho \mathbf{e}_\rho$ we have by equating the two expressions for the $\mathbf{e}_1$-component

$$x^1 = u^\alpha a_\alpha^{\circ 1}$$

and so for the other components. All four are included in the formula

$$x^\sigma = u^\alpha a_\alpha^{\circ \sigma}$$

which expresses the x-components in terms of the u-components.

22. Let us use this formula to express the size-number ν_{OP} in terms of the u-components. We have

$$\begin{aligned}\nu_{OP} &= -\ x^1 x^1 - x^2 x^2 - x^3 x^3 + x^4 x^4 \\ &= -\ u^\alpha a_\alpha^{\circ 1} u^\beta a_\beta^{\circ 1} - u^\alpha a_\alpha^{\circ 2} u^\beta a_\beta^{\circ 2} - u^\alpha a_\alpha^{\circ 3} u^\beta a_\beta^{\circ 3} + u^\alpha a_\alpha^{\circ 4} u^\beta a_\beta^{\circ 4}\end{aligned}$$

As the affix α is an instruction to carry out certain operations, we use a different affix β the second time it occurs in a product in order to avoid confusion.

We can without interfering with the prescribed addition write the right side of the equation in the form

$$u^\alpha u^\beta(-\ a_\alpha^{\circ 1} a_\beta^{\circ 1} - a_\alpha^{\circ 2} a_\beta^{\circ 2} - a_\alpha^{\circ 3} a_\beta^{\circ 3} + a_\alpha^{\circ 4} a_\beta^{\circ 4})$$

This contains 16 terms, obtained by combining the four values of α and the four values of β. The expression 6 of article 20 shows that the 12 terms for which α and β are unequal vanish, that the coefficient of $u^4 u^4$ is $+1$, and the coefficients of $u^1 u^1$ $u^2 u^2$ $u^3 u^3$ are -1. In consequence we have

$$\nu_{OP} = -\,u^1 u^1 - u^2 u^2 - u^3 u^3 + u^4 u^4$$

This shows that the essential quality of any vector $\overrightarrow{OP}$, its size-number, has the same form in terms of its components in the $\mathbf{c}$-lattice as it has in terms of its components in the $\mathbf{e}$-lattice.

New Meaning for the term Perpendicular

23. The **e**-lattice with which we began had its four prime vectors perpendicular to one another in the Euclidian sense as well as conjugate with respect to the contour system, while the **c**-lattice has only conjugacy. At first sight this fact seems to give the **e**-lattice a unique position.

But Euclidian perpendicularity depended on the invariance of the distance between two points, on the invariance of the expression $x^2 + y^2 + z^2$. If we accept the relativity theory of the world which denies that invariance, Euclidian perpendicularity ceases to have any meaning. The uniqueness of the **e**-lattice falls to the ground, and any lattice whose axes are conjugate with respect to the contour system is as good as any other.

The term perpendicular having thus lost its old meaning is free to take on another, and in future we shall use it to mean conjugate with respect to the contour system. Thus a line conjugate to a triplane will be said to be perpendicular to the triplane, two conjugate lines will be called perpendicular, and the **c**-lattice will be called a rectangular lattice.

Orthogonal Projection of a Point-Event

24. The meaning of orthogonal projection in the fourfold is in keeping with the definition of perpendicularity. To project an event P on a triplane we draw through P the line perpendicular to the triplane that is conjugate to the triplane with respect to the contour system. The event M in which this line cuts the triplane is the projection of P.

To project P orthogonally on a biplane B we draw through P the biplane C perpendicular to B, that is to say the biplane C such that every line in C is conjugate to every line in B. The event M in which B and C intersect is the projection of P.

To project P on a line l we draw through P the triplane perpendicular to l, that is conjugate to l, and the projection of P is the event M in which the triplane cuts l.

Parallelism

25. We are now in a position to discuss parallelism. Euclid's way will not do because it requires us to follow straight lines and planes to infinity, and we know that they do not remain straight and plane beyond the local region. We must have a definition that avoids going to infinity.

Suppose T a triplane, and let a vector line l of constant size and normal to T (that is conjugate to T) move so that its foot describes the triplane T. Then its head describes another triplane U which is parallel to T.

The parallel triplanes T U cut any other triplane in parallel biplanes, and cut any biplane in parallel lines.

This discussion assumes the straightness of the lines and the flatness of the biplanes and triplanes, and so is only valid within a limited region around the origin. The question of parallelism for regions beyond is a matter for the higher geometry of the fourfold.

26. We may use alternative definitions of parallelism. Thus in a twofold if a triangle moves in such manner that the base remains in its own straight line, the two sides remain parallel to their original positions.

In a threefold if the base of a tetrahedron remains in a plane and an edge of the base remains in a straight line, all the edges and all the faces have parallel motion.

In the fourfold if one of the bounding tetrahedra of a five-face,* which we may call its base, remains in a triplane while one of the triangles that bound the tetrahedral base remains in a biplane, and a side of the triangle remains in its original straight line, then all the tetrahedra triangles and edges of the five-face move parallel to themselves.

---

* A fiveface is the figure enclosed by five triplanes. It corresponds in the fourfold to the figure in a threefold enclosed by four biplanes, that is to a tetrahedron.

27. The resemblance between a contour trisurface and a hyperboloid of Euclidian space lies in the relation between the coordinates of any point upon it. When we think of a contour trisurface as the locus of points of constant separation from the origin the resemblance is to a sphere, and on this basis alternative names like the following are appropriate for contour trisurfaces and their sections:—

triconicoid	trisphere
trihyperboloid of two sheets	doublecup trisphere
trihyperboloid of one sheet	hourglass trisphere
tricone	trisphere of zero radius
biconicoid	bisphere
bihyperboloid of two sheets	doublecup bisphere
bihyperboloid of one sheet	hourglass bisphere
biellipsoid	closed bisphere
bicone	bisphere of zero radius
conic	circle
hyperbola	circle of two branches (or open circle)
ellipse	closed circle

CHAPTER 4

OBLIQUE LATTICE

1. INSTEAD of the supercubical* lattice in which the four prime vectors have unit size and are at right angles, that is to say conjugate with respect to the contour system, we take any four independent vectors $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ which may have any sizes and any mutual inclinations. We retain the former origin; and the prime vectors $\mathbf{e}_\alpha$ and the new components x^α of a vector $\mathbf{x}$ are homogeneous linear functions of the former prime vectors and components.

The size-number of a vector $\mathbf{x}$ is consequently a homogeneous quadratic function of the components x^α. We write it in the form

$$\nu_{\mathbf{x}} = g_{\alpha\beta} x^\alpha x^\beta$$

in which $g_{\beta\alpha} = g_{\alpha\beta}$ and α β are to be summed for the values 1 2 3 4. There are thus 10 terms, four of the form $g_{11} x^1 x^1$ and six of the form $2g_{12} x^1 x^2$.

2. Let K L be the heads of vectors $\mathbf{k}$ $\mathbf{l}$ at the origin, and let KL cut the contour trisurface of size-number h in the head of vector $\mathbf{x}$. If we put $\mathbf{x} = \kappa\mathbf{k} + \lambda\mathbf{l}$, λ and μ satisfy the equation

$$h = g_{\alpha\beta}(\kappa k^\alpha + \lambda l^\alpha)(\kappa k^\beta + \lambda l^\beta) = \kappa^2 \nu_{\mathbf{k}} + \lambda^2 \nu_{\mathbf{l}} + 2\kappa\lambda g_{\alpha\beta} k^\alpha l^\beta$$

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* A word that in classical language means a three-dimensional figure is converted by the prefix super- to mean the corresponding four-dimensional figure. A supercube has at any corner four edges at right angles and of equal size, and its eight faces are eight equal cubes.

In the last term $2g_{\alpha\beta}k^{\alpha}l^{\beta}$ is a more concise way of writing $g_{\alpha\beta}k^{\alpha}l^{\beta} + g_{\alpha\beta}l^{\alpha}k^{\beta}$, the carrying out of the summation for α β showing the two expressions to have the same value.

$\overrightarrow{OP}$ $\overrightarrow{OQ}$ being the two possible values for **x**, the vectors **k l** will be at right angles, that is to say conjugate with respect to the contour system, if $KLPQ$ is a harmonic range, that is if

$$g_{\alpha\beta}k^{\alpha}l^{\beta} = 0$$

3. The reasoning of article 3,13 and the two following articles follows the same lines in the oblique lattice as in the supercubical lattice. Using the definition of orthogonal projection of article 3,24 we state the results as follows:

(*a*) The perpendicular let fall from the origin on a chord of any contour trisurface bisects the chord.

(*b*) The foot of the perpendicular let fall from the origin on a biplane is the centre of the conic in which the biplane cuts any contour trisurface.

(*c*) The foot of the perpendicular from the origin on a triplane is the centre of the biconicoid in which the triplane cuts any contour trisurface.

4. Let **k l** be two vectors, not perpendicular, with their heads at K L. Draw KM perpendicular to **l**, that is to say, in the biplane **kl** draw KM conjugate to **l** with respect to the conics in which **kl** cuts the contour system. Let **m** denote $\overrightarrow{OM}$ the projection of **k**, and let **p** denote $\overrightarrow{MK}$ the projector.

As **m l** lie along the same line they differ in size only and we put

$$\mathbf{m} = \rho\mathbf{l}$$

Also **p** which we can write as $\mathbf{k} - \mathbf{m}$ or $\mathbf{k} - \rho\mathbf{l}$ is perpendicular to **l** so that

$$0 = g_{\alpha\beta}(k^{\alpha} - \rho l^{\alpha})l^{\beta} = g_{\alpha\beta}k^{\alpha}l^{\beta} - \rho g_{\alpha\beta}l^{\alpha}l^{\beta}$$

We use $\varpi_{\mathbf{kl}}$ to denote $g_{\alpha\beta}k^{\alpha}l^{\beta}$ and have

$$\rho = \varpi_{\mathbf{kl}}/\nu_{\mathbf{l}} \qquad \mathbf{m} = \mathbf{l}\varpi_{\mathbf{kl}}/\nu_{\mathbf{l}}$$

If $\nu_{\mathbf{l}}$ $\varpi_{\mathbf{kl}}$ ρ are all positive we have

$$\sigma_{\mathbf{l}}\sigma_{\mathbf{m}} = \sigma_{\mathbf{l}}\rho\sigma_{\mathbf{l}} = \rho\nu_{\mathbf{l}} = \varpi_{\mathbf{kl}}$$

so that in this case $\varpi_{\mathbf{kl}}$ is the product of the size of **l** and the size of the projection of **k** on **l**. Since $\sigma_{\mathbf{l}}$ $\sigma_{\mathbf{m}}$ are always positive we shall have $\sigma_{\mathbf{l}}\sigma_{\mathbf{m}} = -\varpi_{\mathbf{kl}}$ when $\varpi_{\mathbf{kl}}$ is negative.

Ambiguity of sign is avoided by dealing with size-numbers instead of sizes. We have then in all cases

$$\nu_{\mathbf{l}}\nu_{\mathbf{m}} = \rho^2\nu_{\mathbf{l}}^2 = \varpi_{\mathbf{kl}}^2$$

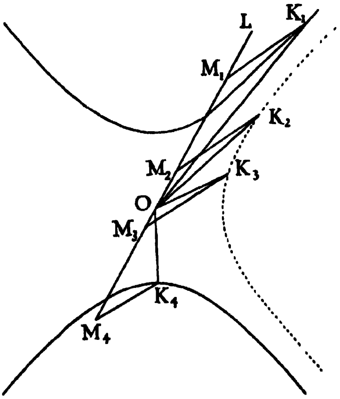


Figure A for article 4,4.

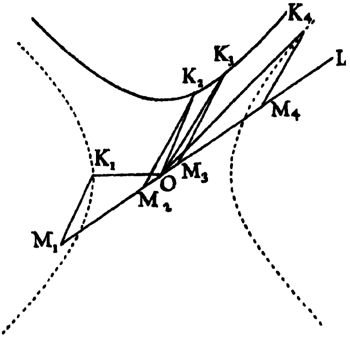


Figure B for article 4,4.

The figures show a number of different cases, **l** being timelike in figure A and spacelike in figure B, and each figure showing four positions for **k**.

5. The quantity $g_{\alpha\beta}k^{\alpha}l^{\beta}$ which we have denoted by $\varpi_{\mathbf{kl}}$ is called the inner product of the vectors **k l**. The symmetry of the expression shows that the product of the size of **k** and the size of the projection of **l** on **k** is equal to the product of the size of **l** and the size of the projection of **k** on **l**.

Another symbol for the inner product $g_{\alpha\beta}k^{\alpha}l^{\beta}$ is $\mathbf{k} \cdot \mathbf{l}$, and by taking the case in which $\mathbf{k} = \mathbf{l}$ we have additional symbols for the size-number $g_{\alpha\beta}k^{\alpha}k^{\beta}$ or $\nu_{\mathbf{k}}$, namely $\varpi_{\mathbf{kk}}$ and $\mathbf{k} \cdot \mathbf{k}$.

6. The coordinates of the head of $\mathbf{e}_1$ are 1 0 0 0 and of $\mathbf{e}_2$ 0 1 0 0, so that

$$\nu_{\mathbf{e}_1} = \mathbf{e}_1 \cdot \mathbf{e}_1 = g_{11}$$

$$\varpi_{e_1 e_2} = \mathbf{e}_1 \cdot \mathbf{e}_2 = g_{12}$$

and so on. Thus the 10 lattice constants $g_{\alpha\beta}$ are expressed in terms of the prime vectors, and the equation

$$g_{\alpha\beta} = \mathbf{e}_{\alpha} \cdot \mathbf{e}_{\beta}$$

combines all ten in one formula.

We note that g_{11} is positive if $\mathbf{e}_1$ is timelike and negative if $\mathbf{e}_1$ is spacelike.

We note also that our whole discussion depends upon the contour trisurfaces and is thus only valid for the "local region." While the quantities $g_{\alpha\beta}$ are constants in the local region we should have to be prepared for variations in them if we went beyond the local region.

7. In the expression

$$\varpi_{\mathbf{kl}} = \mathbf{k} \cdot \mathbf{l} = g_{\alpha\beta}k^{\alpha}l^{\beta}$$

we put $\mathbf{l} = \mathbf{e}_1$ so that l^1 is 1 and l^2 l^3 l^4 are zero, and have

$$\varpi_{\mathbf{k}\mathbf{e}_1} = \mathbf{k} \cdot \mathbf{e}_1 = g_{\alpha 1}k^{\alpha} = g_{11}k^1 + g_{12}k^2 + g_{13}k^3 + g_{14}k^4$$

This quantity we denote by k_1 with the affix placed below. The inner product of $\mathbf{k}$ with $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ gives three similar quantities k_2 k_3 k_4. These four quantities are thus defined by the equation

$$k_{\alpha} = g_{\alpha\omega}k^{\omega}$$

They are, for a reason that will presently appear, called the derived coordinates of the point K and the derived components of the vector **k**.

8. The introduction of these derived quantities gives a new form to $\varpi_{\mathbf{kl}}$ and $\nu_{\mathbf{k}}$. Thus

$$\begin{aligned}\varpi_{\mathbf{kl}} &= g_{\alpha\beta}k^\alpha l^\beta = (g_{1\beta}k^1 + g_{2\beta}k^2 + g_{3\beta}k^3 + g_{4\beta}k^4)l^\beta \\ &= k_\beta l^\beta\end{aligned}$$

and when $\mathbf{l} = \mathbf{k}$ this becomes

$$\nu_{\mathbf{k}} = k_\alpha k^\alpha$$

9. Consider the determinant composed of the lattice constants $g_{\alpha\beta}$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix}$$

We expand in terms of the elements of a row or column, and denote the coefficient of the element $g_{\alpha\beta}$ by $g\, g^{\alpha\beta}$, so that for example

$$g = g(g_{11}g^{11} + g_{12}g^{12} + g_{13}g^{13} + g_{14}g^{14})$$

where

$$gg^{11} = \begin{vmatrix} g_{22} & g_{23} & g_{24} \\ g_{32} & g_{33} & g_{34} \\ g_{42} & g_{43} & g_{44} \end{vmatrix}$$

$$gg^{12} = -\begin{vmatrix} g_{21} & g_{23} & g_{24} \\ g_{31} & g_{33} & g_{34} \\ g_{41} & g_{43} & g_{44} \end{vmatrix} = +\begin{vmatrix} g_{21} & g_{24} & g_{23} \\ g_{31} & g_{34} & g_{33} \\ g_{41} & g_{44} & g_{43} \end{vmatrix}$$

and so on.

10. We denote

$$\begin{vmatrix} g_{\alpha\rho} & g_{\alpha\sigma} & g_{\alpha\tau} \\ g_{\beta\rho} & g_{\beta\sigma} & g_{\beta\tau} \\ g_{\gamma\rho} & g_{\gamma\sigma} & g_{\gamma\tau} \end{vmatrix}$$

by $g_{\alpha\beta\gamma\rho\sigma\tau}$ so that

$$gg^{\alpha\rho} = \begin{vmatrix} g_{\beta\sigma} & g_{\beta\tau} & g_{\beta\phi} \\ g_{\gamma\sigma} & g_{\gamma\tau} & g_{\gamma\phi} \\ g_{\delta\sigma} & g_{\delta\tau} & g_{\delta\phi} \end{vmatrix} = g_{\beta\gamma\delta\sigma\tau\phi}$$

where $\alpha\ \beta\ \gamma\ \delta$ and $\rho\ \sigma\ \tau\ \phi$ are even permutations from 1 2 3 4.

11. The equation of article 9 we can write concisely as

$$g = gg_{1\alpha}g^{1\alpha} \quad \text{or} \quad 1 = g_{1\alpha}g^{1\alpha}$$

and the same equation holds with 2 or 3 or 4 substituted for the affix 1.

Also

$$g_{1\alpha}g^{2\alpha} = g_{11}g^{21} + g_{12}g^{22} + g_{13}g^{23} + g_{14}g^{24}$$

vanishes for g times this quantity is the determinant with the second row replaced by a repetition of the first. And so for $g_{\rho\alpha}g^{\sigma\alpha}$ when ρ and σ differ.

We state all these ten relations in one formula as

$$g_{\rho\alpha}g^{\sigma\alpha} = \delta_{\rho}^{\sigma}$$

where δ_{ρ}^{σ} means 1 when ρ and σ are equal, and means zer when they are unequal. This last set of equations can b taken as the definition of $g^{\alpha\beta}$ since if we solve them for g^{α} we obtain the values given in article 9

12. We had in article 7 the equation

$$x_{\alpha} = g_{\alpha\rho}x^{\rho}$$

to express the derived components x_{α} of a vector **x** in term of the original components x^{α}. We can also express th original components in terms of the derived ones. We for the expression $g^{\beta\alpha}x_{\alpha}$, that is to say we take x_1 and multipl it by $g^{\beta 1}$, and treat x_2 x_3 x_4 similarly and add. The expression $g_{1\rho}x^\rho$ has four terms so that on the right side of

$$g^{\beta\alpha}x_\alpha = g^{\beta\alpha}g_{\alpha\rho}x^\rho$$

there are 16 terms. We take together the four terms that contain x^1 namely $g^{\beta\alpha}g_{\alpha 1}$ which is δ^β_1 which is unity if $\beta = 1$ and is zero for any other value of β.

Treating the other terms similarly we have

$$g^{\beta\alpha}x_\alpha = \delta^\beta_\rho x^\rho$$

The expression on the right bids us sum four terms for $\rho = 1\ 2\ 3\ 4$. The one for which $\rho = \beta$ is unity and the rest zero, so that

$$g^{\beta\alpha}x_\alpha = x^\beta$$

that is to say we have expressed x^1 x^2 x^3 x^4 in terms of x_1 x_2 x_3 x_4.

13. **Exercise.**—In an oblique lattice $\mathbf{e}_{1234}$ is there any restriction on the number of the prime vectors that may be timelike, or on the number that may be spacelike?

CHAPTER 5

THE DERIVED LATTICE IN A TWOFOLD

1. As a manifold of two dimensions we shall discuss the biplane containing the two vectors $\mathbf{e}_1$ $\mathbf{e}_2$. A vector $\mathbf{x}$ lying in this twofold has the components x^3 x^4 equal to zero, so that its size-number is

$$\nu_{\mathbf{x}} = g_{11}x^1x^1 + 2g_{12}x^1x^2 + g_{22}x^2x^2$$

The determinant $\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$ which we denote by g_{1212} we expand, denoting the coefficient of the element $g_{\alpha\beta}$ by $gg^{\alpha\beta}$. Thus the expansions are

$$g_{1212}(g_{11}g^{11} + g_{12}g^{12}) \quad \text{and} \quad g_{1212}(g_{21}g^{21} + g_{22}g^{22})$$

in which

$$g_{1212}g^{11} = g_{22} \qquad g_{1212}g^{12} = -g_{12} \qquad g_{1212}g^{22} = g_{11}$$

We note that these values give

$$g_{\alpha\omega}g^{\beta\omega} = \delta^{\beta}_{\alpha}$$

the summation being for the values 1 2 of ω.

2. We define two new vectors $\mathbf{e}^1$ $\mathbf{e}^2$ with their affixes above by the equations

$$\mathbf{e}^{\alpha} = g^{\alpha\omega}\mathbf{e}_{\omega}$$

or written in full

$$\mathbf{e}^1 = g^{11}\mathbf{e}_1 + g^{12}\mathbf{e}_2 \qquad \mathbf{e}^2 = g^{21}\mathbf{e}_1 + g^{22}\mathbf{e}_2$$

Along $\mathbf{e}_1$ we mark off $\overrightarrow{OP} = g^{11}\mathbf{e}_1$ and through P we draw $\overrightarrow{PE^1}$ parallel to $\mathbf{e}_2$ and equal to $g^{12}\mathbf{e}_2$, and so find E^1 the head of $\mathbf{e}^1$. We also let the perpendicular E^1M fall on $\mathbf{e}_1$. Then

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = \overrightarrow{OM} \cdot \mathbf{e}_1 = \overrightarrow{OP} \cdot \mathbf{e}_1 + \overrightarrow{PM} \cdot \mathbf{e}_1$$

The first term is $g^{11}\mathbf{e}_1 \cdot \mathbf{e}_1$ or $g^{11}g_{11}$. The second term $\overrightarrow{PM} \cdot \mathbf{e}_1$ is the same as $\overrightarrow{PE^1} \cdot \mathbf{e}_1$ or $g^{12}\mathbf{e}_2 \cdot \mathbf{e}_1$ that is $g^{12}g_{12}$. Hence

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = g^{11}g_{11} + g^{12}g_{12} = 1$$

In the same way we find

$$\mathbf{e}^1 \cdot \mathbf{e}_2 = g^{11}g_{21} + g^{12}g_{22} = 0$$

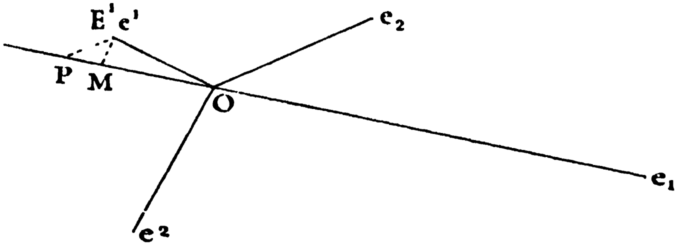


Figure for article 5,2.

Thus $\mathbf{e}^1$ is perpendicular to $\mathbf{e}_2$ and of such size as to make its inner product with $\mathbf{e}_1$ unity.

Similarly $\mathbf{e}^2$ is perpendicular to $\mathbf{e}_1$ and has unit inner product with $\mathbf{e}_2$.

3. **Exercise.**—Use the expression $g_{\alpha\beta}k^\alpha l^\beta$ for the inner product of $\mathbf{k}$ and $\mathbf{l}$ to find the inner products $\mathbf{e}^1 . \mathbf{e}_1$ $\mathbf{e}^1 . \mathbf{e}_2$.

4. The height of a parallelogram is defined as the line terminated by, and normal to, a pair of parallel sides, normalcy meaning conjugacy with respect to the contour system. The figures show the two heights $\mathbf{h}_1$ $\mathbf{h}_2$ of the parallelogram $\mathbf{e}_1\mathbf{e}_2$. In figure A the vectors $\mathbf{e}_1$ $\mathbf{e}_2$ are time-

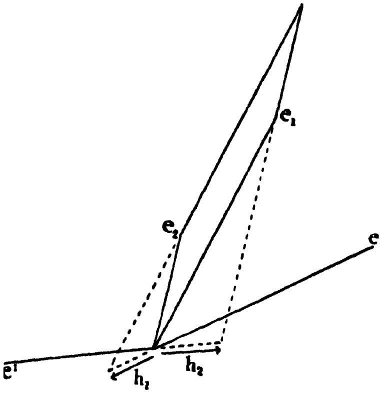


Figure A for article 5,3.

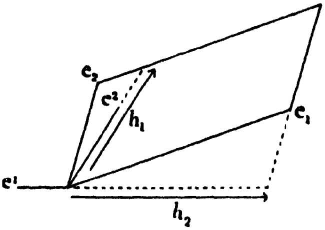


Figure B for article 5,3.

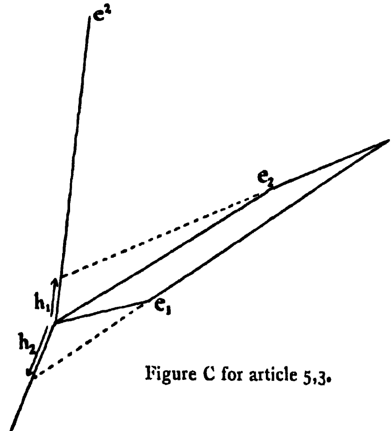


Figure C for article 5,3.

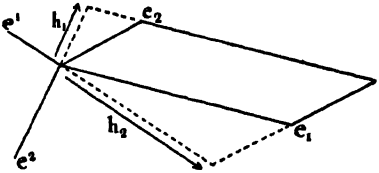


Figure D for article 5,3.

like and $\mathbf{e}^1$ $\mathbf{e}^2$ spacelike, in figure B $\mathbf{e}_1$ is spacelike and $\mathbf{e}_2$ timelike, $\mathbf{e}^1$ spacelike and $\mathbf{e}^2$ timelike, in figure C $\mathbf{e}_1$ $\mathbf{e}_2$ are spacelike and $\mathbf{e}^1$ $\mathbf{e}^2$ timelike, in figure D $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}^1$ $\mathbf{e}^2$ are all spacelike.

The size $\sigma_{\mathbf{a}}$ and size-number $\nu_{\mathbf{a}}$ of the area vector $\mathbf{e}_1\mathbf{e}_2$ are defined to be $\sigma_{\mathbf{e}_1}\sigma_{\mathbf{h}_1}$ and $\nu_{\mathbf{e}_1}\nu_{\mathbf{h}_1}$ where $\sigma_{\mathbf{e}_1}$ $\sigma_{\mathbf{h}_1}$ and $\nu_{\mathbf{e}_1}$ $\nu_{\mathbf{h}_1}$ are the sizes and size-numbers of the vectors $\mathbf{e}_1$ $\mathbf{h}_1$.

Now $\mathbf{h}_1$ has the form $\lambda\mathbf{e}^2$ so that

$$1 = \mathbf{e}_2 \bullet \mathbf{e}^2 = \mathbf{h}_1 \bullet \mathbf{e}^2 = \lambda\mathbf{e}^2 \bullet \mathbf{e}^2 = \lambda g^{22} = \lambda g_{11}/g_{1212}$$

and

$$\mathbf{h}_1 = \mathbf{e}^2/g^{22}$$

$$\nu_{\mathbf{h}_1} = 1/g^{22} = g_{1212}/g_{11}$$

Hence

$$\nu_{\mathbf{a}} = \nu_{\mathbf{e}_1}\nu_{\mathbf{h}_1} = g_{1212}$$

5. The biplane $\mathbf{e}_{12}$ may or may not cut the contour tricone in real lines. In the former case it contains both timelike and spacelike vectors and is called a mixed biplane. In the latter case it contains only spacelike vectors and is called a spacelike biplane.

The result of article 4 shows that if g_{1212} is positive $\mathbf{e}_1$ $\mathbf{h}_1$ must be of the same kind, and therefore both spacelike; if it is negative they are of different kinds, one timelike and one spacelike.

Hence if g_{1212} is negative the biplane is mixed and

$$\sigma_{\mathbf{a}} = \sqrt{-g_{1212}}$$

and if it is positive the biplane is spacelike and

$$\sigma_{\mathbf{a}} = \sqrt{+g_{1212}}$$

6. The biplane $\mathbf{e}_{12}$ cuts the contour tricone along any vector $\mathbf{x}$ for which $g_{\alpha\beta}x^\alpha x^\beta$ $(\alpha\ \beta = 1\ 2)$ is zero, so that it

cuts in real lines if the equation $g_{\alpha\beta}x^\alpha x^\beta = 0$ has real roots, that is to say if g_{1212} is negative. That is to say $\mathbf{e}_{12}$ is spacelike if g_{1212} is positive, mixed if it is negative; in agreement with the preceding paragraph.

7. We saw in article 4 that the size-number of $\mathbf{a} = \mathbf{e}_1\mathbf{e}_2$ is

$$\nu_{\mathbf{e}_1\mathbf{e}_2} = g_{1212} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$$

This determinant can be expressed as $\begin{vmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{e}_2 \end{vmatrix}$ and we can take the prime vectors to coincide with any two vectors **k** **l**, so that we may state the general result:

$$\nu_{\mathbf{kl}} = \begin{vmatrix} \mathbf{k} \cdot \mathbf{k} & \mathbf{k} \cdot \mathbf{l} \\ \mathbf{l} \cdot \mathbf{k} & \mathbf{l} \cdot \mathbf{l} \end{vmatrix}$$

8. **Exercise.**—Find the relation between $\nu_{\mathbf{l}}$ $\nu_{\mathbf{m}}$ $\varpi_{\mathbf{lm}}$ for which the biplane of the area vector **lm** contains timelike vectors, the vectors **l** **m** being supposed spacelike.

9. **Exercise.**—Show that the size-number of the area vector **lm** is positive if the area is spacelike and negative if the area is of mixed character.

10. Now consider any vector $\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$. The derived coordinates x_1 x_2 are defined by

$$\begin{aligned} x_\alpha &= \mathbf{x} \cdot \mathbf{e}_\alpha \\ &= x^1\mathbf{e}_1 \cdot \mathbf{e}_\alpha + x^2\mathbf{e}_2 \cdot \mathbf{e}_\alpha \\ &= g_{\alpha 1}x^1 + g_{\alpha 2}x^2 \\ &= g_{\alpha\rho}x^\rho \end{aligned} \qquad (1)$$

From this

$$\begin{aligned} g^{\sigma\alpha}x_\alpha &= g^{\sigma\alpha}g_{\alpha\rho}x^\rho \\ &= \delta^\sigma_\rho x^\rho \\ &= x^\sigma \end{aligned} \qquad (2)$$

Now consider $x_\alpha\mathbf{e}^\alpha = x_1\mathbf{e}^1 + x_2\mathbf{e}^2$.

Expressing $\mathbf{e}^1\ \mathbf{e}^2$ in terms of $\mathbf{e}_1\ \mathbf{e}_2$ we have

$$x_\alpha \mathbf{e}^\alpha = x_\alpha g^{\alpha\beta} \mathbf{e}_\beta$$

And equation 2 gives $x_\alpha g^{\alpha\beta} = x^\beta$ so that

$$x_\alpha \mathbf{e}^\alpha = x^\beta \mathbf{e}_\beta$$

or written in full

$$x_1 \mathbf{e}^1 + x_2 \mathbf{e}^2 = x^1 \mathbf{e}_1 + x^2 \mathbf{e}_2 = \mathbf{x}$$

that is to say the journey from the origin to the head of the vector $\mathbf{x}$ may be made by going $x_1 \mathbf{e}^1$ along $\mathbf{e}^1$ and $x_2 \mathbf{e}^2$ along $\mathbf{e}^2$. And this means that if we treat $\mathbf{e}^1\ \mathbf{e}^2$ as the prime vectors of a reference lattice, the components of the vector $\mathbf{x}$ in this lattice are $x_1\ x_2$.

We call the lattice $\mathbf{e}^{12}$ the derived lattice in the twofold and $\mathbf{e}_{12}$ the original lattice. The two together we call the reference system in the twofold.

11. The original and derived lattices are completely reciprocal to one another. Thus the size-number is

$$\begin{aligned} \nu_{\mathbf{x}} &= g_{\alpha\beta} x^\alpha x^\beta \\ &= g_{\alpha\beta} g^{\alpha\theta} x_\theta g^{\beta\phi} x_\phi \\ &= \delta^\theta_\beta x_\theta g^{\beta\phi} x_\phi \\ &= g^{\theta\phi} x_\theta x_\phi \end{aligned}$$

which is the appropriate expression in the derived lattice.

Also if we denote the derived determinant

$$\begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix}$$

by g^{1212}, we see that $g^{1212} g_{1212} = 1$. We can define $g_{\alpha\beta}$ by the expansion of this determinant; thus

$$g^{1212} = g^{1212}\,(g^{11} g_{11} + g^{12} g_{12}) = g^{1212}(g^{21} g_{21} + g^{22} g_{22})$$

which gives the appropriate values to $g_{11}\ \ g_{12}\ \ g_{22}$.

CHAPTER 6.

THE DERIVED LATTICE IN A THREEFOLD

1. THE three prime vectors $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ determine a triplane which we shall now consider as a threefold. For any vector $\mathbf{x}$ lying in this threefold the component x^4 is zero, so that its size-number is given by

$$\nu_{\mathbf{x}} = g_{\rho\sigma}x^{\rho}x^{\sigma}$$

in which ρ σ take the values 1 2 3.

We denote the determinant

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$

by g_{123123}. We denote the coefficient of $g_{\alpha\beta}$ in an expansion of the determinant by $g_{123123}g^{\alpha\beta}$. Thus for example

$$g_{123123} = g_{123123}(g_{11}g^{11} + g_{12}g^{12} + g_{13}g^{13}) = g_{123123}g_{1\omega}g^{1\omega}$$

where

$$g_{123123}g^{11} = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} \qquad g_{123123}g^{12} = \begin{vmatrix} g_{23} & g_{21} \\ g_{33} & g_{31} \end{vmatrix}$$

and generally, α β γ and ρ σ τ being even permutations of 1 2 3,

$$g_{123123}g^{\alpha\rho} = \begin{vmatrix} g_{\beta\sigma} & g_{\beta\tau} \\ g_{\gamma\sigma} & g_{\gamma\tau} \end{vmatrix}$$

We use $g_{\beta\gamma\sigma\tau}$ to denote this last determinant, and the equation takes the form

$$g_{123123}g^{\alpha\rho} = g_{\beta\gamma\sigma\tau}$$

The expansion of g_{123123} shows that $g_{\alpha\omega}g^{\beta\omega}$ is unity when α and β are equal. When they are unequal the expression comes from the expansion of a determinant with two identical rows, and is therefore zero. All cases are therefore covered in the statement

$$g_{\alpha\omega}g^{\beta\omega} = \delta_{\alpha}^{\beta}$$

2. We now introduce three vectors $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ defined by

$$\mathbf{e}^{\alpha} = g^{\alpha\omega}\mathbf{e}_{\omega}$$

in which we sum for the values 1 2 3 of ω and α takes the values 1 2 3. The new vectors being linear functions of $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ will lie in the same threefold $\mathbf{e}_{123}$.

3. For the inner product of two vectors $\mathbf{k}$ $\mathbf{l}$ which lie in the threefold and have no $\mathbf{e}_4$-component the expression becomes

$g_{\alpha\beta}k^{\alpha}l^{\beta}$ summed for the values 1 2 3 of α and β

We make $\mathbf{k}$ equal to $\mathbf{e}^1$ which has the components g^{11} g^{12} g^{13} and $\mathbf{l}$ equal to $\mathbf{e}_2$ which has components 0 1 0. Then for the inner product $\mathbf{e}^1 \cdot \mathbf{e}_2$ we have

$$g_{\alpha 2}k^{\alpha} = g_{\alpha 2}g^{1\alpha} = 0$$

Taking in this way the inner product of $\mathbf{e}^1$ with each of the prime vectors $\mathbf{e}_{\alpha}$ we find that $\mathbf{e}^1$ is normal to $\mathbf{e}_2$ and $\mathbf{e}_3$ and has inner product unity with $\mathbf{e}_1$.

In the same way we see that $\mathbf{e}^2$ has inner product unity with $\mathbf{e}_2$ and is normal to $\mathbf{e}_3$ and $\mathbf{e}_1$, and that $\mathbf{e}^3$ has inner product unity with $\mathbf{e}_3$ and is normal to $\mathbf{e}_1$ and $\mathbf{e}_2$.

4. The reasoning of article 3 can be put in a somewhat different form. Thus

$$\mathbf{e}^{\alpha} \cdot \mathbf{e}_{\beta} = (g^{\alpha\omega}\mathbf{e}_{\omega}) \cdot \mathbf{e}_{\beta} = g^{\alpha\omega}(\mathbf{e}_{\omega} \cdot \mathbf{e}_{\beta}) = g^{\alpha\omega}g_{\beta\omega} = \delta^{\alpha}_{\beta}$$

This properly understood gives the reasoning of article 3, and it also treats all nine cases at once.

5. Since $\mathbf{e}^3$ is normal to $\mathbf{e}_{12}$, that is conjugate to it with respect to the contour system, the height $\mathbf{h}$ of the triped $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ when we look on $\mathbf{e}_1\mathbf{e}_2$ as base lies along $\mathbf{e}^3$.

We define the size $\sigma_{\mathbf{v}}$ and size-number $\nu_{\mathbf{v}}$ of the triped as being $\sigma_{\mathbf{a}}\sigma_{\mathbf{h}}$ and $\nu_{\mathbf{a}}\nu_{\mathbf{h}}$, where $\sigma_{\mathbf{a}}$ $\sigma_{\mathbf{h}}$ and $\nu_{\mathbf{a}}$ $\nu_{\mathbf{h}}$ are the sizes and size-numbers of the parallelogram $\mathbf{e}_1\mathbf{e}_2$ and the height $\mathbf{h}$.

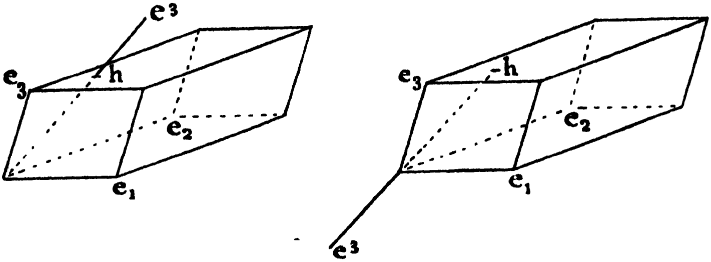


Figure for article 6,5.

Since $\mathbf{h}$ has the form $\lambda\mathbf{e}^3$ and is the projection of $\mathbf{e}_3$ upon $\mathbf{e}^3$ we have

$$1 = \mathbf{e}_3 \cdot \mathbf{e}^3 = \mathbf{h} \cdot \mathbf{e}^3 = \lambda\mathbf{e}^3 \cdot \mathbf{e}^3 = \lambda g^{33}$$

and by the aid of the definition of g^{33} given in article 1

$$\nu_{\mathbf{h}} = \lambda^2\mathbf{e}^3 \cdot \mathbf{e}^3 = \lambda^2 g^{33} = 1/g^{33} = g_{123123}/g_{1212}$$
$$\nu_{\mathbf{v}} = \nu_{\mathbf{a}}\nu_{\mathbf{h}} = g_{123123}$$

6. The triplane $\mathbf{e}_{123}$ may cut the contour tricone in a real bicone or it may not, and is called a mixed or a spacelike

triplane accordingly. The three directions $\mathbf{e}_1$, the perpendicular to $\mathbf{e}_1$ in the biplane $\mathbf{e}_{12}$ and $\mathbf{h}$ are mutually perpendicular. In a mixed triplane by article 3,19 one only of these three perpendicular directions is timelike, so that g_{123123} is positive, and

$$\sigma_{\mathbf{v}} = \sqrt{g_{123123}}$$

In a spacelike triplane, all three perpendicular directions are spacelike, g_{123123} is negative, and

$$\sigma_{\mathbf{v}} = \sqrt{-g_{123123}}$$

7. The condition of a real intersection between $\mathbf{e}_{123}$ and the tricone is the condition that $g_{\alpha\beta}x^\alpha x^\beta$ $(\alpha\ \beta = 1\ 2\ 3)$ should have zero values with x^α finite. Thus we see that the condition that $g_{\alpha\beta}x^\alpha x^\beta$ should have zero values with x^α finite is that g_{123123} is positive.

8. By expressing $g_{\alpha\beta}$ in the form $\mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ we have (from article 5) for the size-number of the triped

$$\nu_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3} = \begin{vmatrix} \mathbf{e}_1\cdot\mathbf{e}_1 & \mathbf{e}_1\cdot\mathbf{e}_2 & \mathbf{e}_1\cdot\mathbf{e}_3 \\ \mathbf{e}_2\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{e}_2 & \mathbf{e}_2\cdot\mathbf{e}_3 \\ \mathbf{e}_3\cdot\mathbf{e}_1 & \mathbf{e}_3\cdot\mathbf{e}_2 & \mathbf{e}_3\cdot\mathbf{e}_3 \end{vmatrix}$$

and as any three independent vectors can be chosen as the prime vectors the result holds for any vectors $\mathbf{l}\ \mathbf{m}\ \mathbf{n}$, that is to say

$$\nu_{\mathbf{lmn}} = \begin{vmatrix} \mathbf{l}\cdot\mathbf{l} & \mathbf{l}\cdot\mathbf{m} & \mathbf{l}\cdot\mathbf{n} \\ \mathbf{m}\cdot\mathbf{l} & \mathbf{m}\cdot\mathbf{m} & \mathbf{m}\cdot\mathbf{n} \\ \mathbf{n}\cdot\mathbf{l} & \mathbf{n}\cdot\mathbf{m} & \mathbf{n}\cdot\mathbf{n} \end{vmatrix}$$

9. Consider any vector

$$\mathbf{x} = x^\alpha \mathbf{e}_\alpha = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$$

repetition of an affix directing summation for the values 1 2 3. The derived components are given by

$$x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha = g_{\alpha\rho} x^\rho$$

From this we find

$$g^{\sigma\alpha} x_\alpha = g^{\sigma\alpha} g_{\rho\alpha} x^\rho = \delta^\sigma_\rho x^\rho = x^\sigma$$

and

$$x_\alpha \mathbf{e}^\alpha = g_{\alpha\rho} x^\rho \mathbf{e}^\alpha = x^\rho \mathbf{e}_\rho = \mathbf{x}$$

Thus x_α are the ordinary components of vector $\mathbf{x}$ in a lattice of which $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ are the prime vectors. Accordingly we call this the derived lattice of $\mathbf{e}_{123}$.

10. Further

$$\mathbf{e}^\alpha \cdot \mathbf{e}^\beta = g^{\alpha\rho} \mathbf{e}_\rho \cdot g^{\beta\sigma} \mathbf{e}_\sigma = g^{\alpha\rho} g^{\beta\sigma} g_{\rho\sigma} = \delta^\alpha_\sigma g^{\beta\sigma} = g^{\alpha\beta}$$

so that the $g^{\alpha\beta}$ have the appropriate meaning in the derived lattice; and

$$g^{\alpha\beta} x_\alpha x_\beta = x_\alpha x^\alpha = g_{\alpha\rho} x^\rho x^\alpha = \nu_\mathbf{x}$$

so that the size-number has the appropriate expression.

We solve the three equations

$$g_{1\alpha} g^{1\alpha} = 1 \qquad g_{1\alpha} g^{2\alpha} = 0 \qquad g_{1\alpha} g^{3\alpha} = 0$$

for g_{11} g_{12} g_{13} and find

$$\begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix} g_{11} = \begin{vmatrix} g^{22} & g^{23} \\ g^{32} & g^{33} \end{vmatrix} \qquad \text{etc.}$$

so that

$$g^{123123} g_{\alpha\rho} = g^{\beta\gamma\sigma\tau}$$

where α β γ and ρ σ τ are even permutations from 1 2 3.

Thus everything is reciprocal between the original lattice $\mathbf{e}_{123}$ and the derived lattice $\mathbf{e}^{123}$.

11. **Exercise.**—Under what circumstances do the prime vectors $\mathbf{e}^1$ $\mathbf{e}^2$ of the derived lattice of $\mathbf{e}_{123}$ coincide with the two prime vectors of the derived lattice of $\mathbf{e}_{12}$?

CHAPTER 7

THE DERIVED LATTICE IN THE FOURFOLD

1. In the fourfold we take four vectors $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ $\mathbf{e}^4$ defined by

$$\mathbf{e}^\alpha = g^{\alpha\rho}\mathbf{e}_\rho$$

in which $g^{\alpha\beta}$ are the constants defined in article 4,9
Then.

$$\mathbf{e}^\alpha \cdot \mathbf{e}_\beta = g^{\alpha\rho}\mathbf{e}_\rho \cdot \mathbf{e}_\beta = g^{\alpha\rho}g_{\beta\rho} = \delta^\alpha_\beta$$

so that $\mathbf{e}^1$ is normal to the triplane $\mathbf{e}_{234}$ and of such size that $\mathbf{e}^1 \cdot \mathbf{e}_1$ is unity. And so for $\mathbf{e}^2$ $\mathbf{e}^3$ $\mathbf{e}^4$.

Also

$$\mathbf{e}^\alpha \cdot \mathbf{e}^\beta = g^{\alpha\rho}\mathbf{e}_\rho \cdot g^{\beta\sigma}\mathbf{e}_\sigma = g^{\alpha\rho}g^{\beta\sigma}g_{\rho\sigma} = \delta_\sigma g^{\beta\sigma} = g^{\alpha\beta}$$

2. **Exercise.**—In what circumstances do the three vectors $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ of the derived lattice of $\mathbf{e}_{1234}$ coincide with the three prime vectors of the derived lattice of $\mathbf{e}_{123}$?

3. Taking the triped $\mathbf{v} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ as the base of the tetraped $f \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$ we define the height to be the vector $\mathbf{h}$ which is normal to $\mathbf{v}$, has its foot at O in $\mathbf{v}$, and has its head in the triped obtained by a parallel shift of $\mathbf{v}$ along $\mathbf{e}_4$; which triped we may call the top of the tetraped.

We define the size σ_f and size-number ν_f of the tetraped to be $\sigma_{\mathbf{v}}\sigma_{\mathbf{h}}$ and $\nu_{\mathbf{v}}\nu_{\mathbf{h}}$ where $\sigma_{\mathbf{v}}$ $\sigma_{\mathbf{h}}$ and $\nu_{\mathbf{v}}$ $\nu_{\mathbf{h}}$ are the sizes and size-numbers of the triped $\mathbf{v}$ and the height $\mathbf{h}$.

Since $\mathbf{h}$ and $\mathbf{e}^4$ are both normal to the triplane $\mathbf{e}_{123}$ we have

$$\mathbf{h} = \lambda\mathbf{e}^4$$

and since **h** is the orthogonal projection of $\mathbf{e}_4$ on $\mathbf{e}^4$

$$1 = \mathbf{e}_4 \cdot \mathbf{e}^4 = \mathbf{h} \cdot \mathbf{e}^4 = \lambda \mathbf{e}^4 \cdot \mathbf{e}^4 = \lambda g^{44}$$

Hence with the aid of the definition of g^{44} given in article 4,10

$$\nu_{\mathbf{h}} = \lambda^2 \mathbf{e}^4 \cdot \mathbf{e}^4 = \lambda^2 g^{44} = 1/g^{44} = g/g_{123123}$$

and

$$\nu_f = \nu_{\mathbf{h}} \nu_{\mathbf{v}} = g g_{123123}/g_{123123} = g$$

4. Let **k** be the height of the vector area $\mathbf{a} \equiv \mathbf{e}_1\mathbf{e}_2$ when $\mathbf{e}_1$ is taken as base, and let **l** be the height of the vector volume

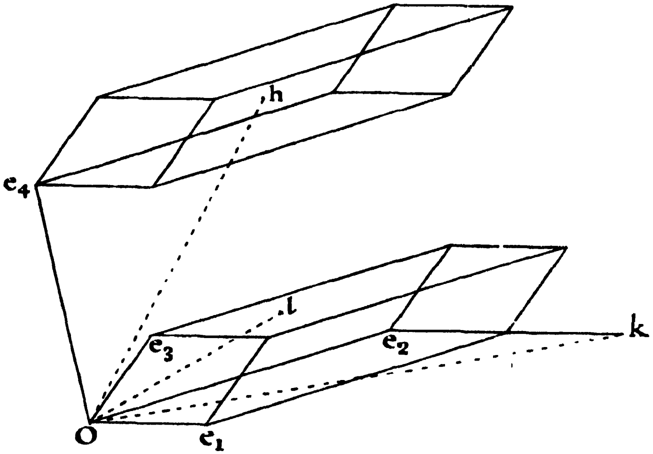


Figure for article 7,4.

$\mathbf{v} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ when vector area $\mathbf{e}_1\mathbf{e}_2$ is taken as base. Then we have

$$\nu_{\mathbf{a}} = \nu_{\mathbf{e}_1} \nu_{\mathbf{k}}$$

$$\nu_{\mathbf{v}} = \nu_{\mathbf{a}} \nu_{\mathbf{l}} = \nu_{\mathbf{e}_1} \nu_{\mathbf{k}} \nu_{\mathbf{l}}$$

so that

$$\nu_f = \nu_{\mathbf{v}} \nu_{\mathbf{h}} = \nu_{\mathbf{e}_1} \nu_{\mathbf{k}} \nu_{\mathbf{l}} \nu_{\mathbf{h}}$$

Also **k** is normal to $\mathbf{e}_1$, **l** is normal to $\mathbf{e}_{12}$ and so to $\mathbf{e}_1$ and **k**, and **h** is normal to $\mathbf{e}_{123}$ and so to $\mathbf{e}_1$ **k** **l**. That is

$\mathbf{e}_1\ \mathbf{k}\ \mathbf{l}\ \mathbf{h}$ form a set of rectangular vectors. Therefore one of them is timelike and the rest spacelike, so that ν_f is negative, and

$$\sigma_f = \sqrt{-\nu_f} = \sqrt{-g}$$

5. If we express $g_{\alpha\beta}$ as $\mathbf{e}_\alpha \bullet \mathbf{e}_\beta$ we have for the size-number of the tetraped $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$

$$\nu_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4} = \begin{vmatrix} \mathbf{e}_1 \bullet \mathbf{e}_1 & \mathbf{e}_1 \bullet \mathbf{e}_2 & \mathbf{e}_1 \bullet \mathbf{e}_3 & \mathbf{e}_1 \bullet \mathbf{e}_4 \\ \mathbf{e}_2 \bullet \mathbf{e}_1 & \mathbf{e}_2 \bullet \mathbf{e}_2 & \mathbf{e}_2 \bullet \mathbf{e}_3 & \mathbf{e}_2 \bullet \mathbf{e}_4 \\ \mathbf{e}_3 \bullet \mathbf{e}_1 & \mathbf{e}_3 \bullet \mathbf{e}_2 & \mathbf{e}_3 \bullet \mathbf{e}_3 & \mathbf{e}_3 \bullet \mathbf{e}_4 \\ \mathbf{e}_4 \bullet \mathbf{e}_1 & \mathbf{e}_4 \bullet \mathbf{e}_2 & \mathbf{e}_4 \bullet \mathbf{e}_3 & \mathbf{e}_4 \bullet \mathbf{e}_4 \end{vmatrix}$$

and as we may take any four independent vectors as the prime vectors we have the result for the size-number of the tetraped of which $\mathbf{l}\ \mathbf{m}\ \mathbf{n}\ \mathbf{p}$ are four concurrent edges

$$\nu_{\mathbf{lmnp}} = \begin{vmatrix} \mathbf{l} \bullet \mathbf{l} & \mathbf{l} \bullet \mathbf{m} & \mathbf{l} \bullet \mathbf{n} & \mathbf{l} \bullet \mathbf{p} \\ \mathbf{m} \bullet \mathbf{l} & \mathbf{m} \bullet \mathbf{m} & \mathbf{m} \bullet \mathbf{n} & \mathbf{m} \bullet \mathbf{p} \\ \mathbf{n} \bullet \mathbf{l} & \mathbf{n} \bullet \mathbf{m} & \mathbf{n} \bullet \mathbf{n} & \mathbf{n} \bullet \mathbf{p} \\ \mathbf{p} \bullet \mathbf{l} & \mathbf{p} \bullet \mathbf{m} & \mathbf{p} \bullet \mathbf{n} & \mathbf{p} \bullet \mathbf{p} \end{vmatrix}$$

6. In the particular case in which the biplanes $\mathbf{lm}$ $\mathbf{np}$ are at right angles, that is to say $\mathbf{l}\ \mathbf{m}$ are both perpendicular to $\mathbf{n}$ and to $\mathbf{p}$, we have

$$\nu_{\mathbf{lmnp}} = \begin{vmatrix} \mathbf{l} \bullet \mathbf{l} & \mathbf{l} \bullet \mathbf{m} & 0 & 0 \\ \mathbf{m} \bullet \mathbf{l} & \mathbf{m} \bullet \mathbf{m} & 0 & 0 \\ 0 & 0 & \mathbf{n} \bullet \mathbf{n} & \mathbf{n} \bullet \mathbf{p} \\ 0 & 0 & \mathbf{p} \bullet \mathbf{n} & \mathbf{p} \bullet \mathbf{p} \end{vmatrix} = \nu_{\mathbf{lm}}\,\nu_{\mathbf{np}}$$

so that also

$$\nu_{\mathbf{e}^1\mathbf{e}^2\mathbf{e}_3\mathbf{e}_4} = \nu_{\mathbf{e}^1\mathbf{e}^2}\nu_{\mathbf{e}_3\mathbf{e}_4} = g^{1212}g_{3434}$$

7. There is complete reciprocity between the lattice system based on $\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\ \mathbf{e}_4$ and that based on $\mathbf{e}^1\ \mathbf{e}^2\ \mathbf{e}^3\ \mathbf{e}^4$.

In the former case we defined the components of a vector $\mathbf{x}$ by

$$\mathbf{x} = x^{\alpha}\mathbf{e}_{\alpha}$$

the size-number by

$$\nu_{\mathbf{x}} = g_{\alpha\beta}x^{\alpha}x^{\beta}$$

and the constants $g^{\alpha\beta}$ by

$$g^{\alpha\rho}g_{\beta\rho} = \delta^{\alpha}_{\beta}$$

From them we defined $\mathbf{e}^{\alpha}$ x_{α} by

$$x_{\alpha} = g_{\alpha\rho}x^{\rho}$$
$$\mathbf{e}^{\alpha} = g^{\alpha\rho}\mathbf{e}_{\rho}$$

and deduced

$$\mathbf{x} = x_{\alpha}\mathbf{e}^{\alpha}$$
$$\nu_{\mathbf{x}} = g^{\alpha\beta}x_{\alpha}x_{\beta}$$

We can begin with the lattice system $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ $\mathbf{e}^4$, define the components of a vector by

$$\mathbf{x} = x_{\alpha}\mathbf{e}^{\alpha}$$

and the size-number by

$$\nu_{\mathbf{x}} = g^{\alpha\beta}x_{\alpha}x_{\beta}$$

with $g_{\alpha\beta}$ defined by

$$g_{\alpha\rho}g^{\beta\rho} = \delta^{\beta}_{\alpha}$$

From these we change to the other lattice by the definitions

$$x^{\alpha} = g^{\alpha\rho}x_{\rho}$$
$$\mathbf{e}_{\alpha} = g_{\alpha\rho}\mathbf{e}^{\rho}$$

and deduce

$$\mathbf{x} = x^{\alpha}\mathbf{e}_{\alpha}$$
$$\nu_{\mathbf{x}} = g_{\alpha\beta}x^{\alpha}x^{\beta}$$

We thus have exactly the same equations, no matter from which end we begin.

CHAPTER 8

INCLINATION BETWEEN TWO LINE VECTORS

1. We take two vectors $\overrightarrow{OL} \equiv \mathbf{l}$ and $\overrightarrow{OM} \equiv \mathbf{m}$, and drop the perpendicular MQ from M on OL. We can use the ratios of the sizes or size-numbers of OQ QM OM to one another as measures of the inclination of the two vectors. We know from articles 4,4 and 5,7 that

$$\nu_{OQ} = \varpi^2_{\mathbf{lm}}/\nu_{\mathbf{l}} \qquad \nu_{QM} = \nu_{\mathbf{lm}}/\nu_{\mathbf{l}}$$

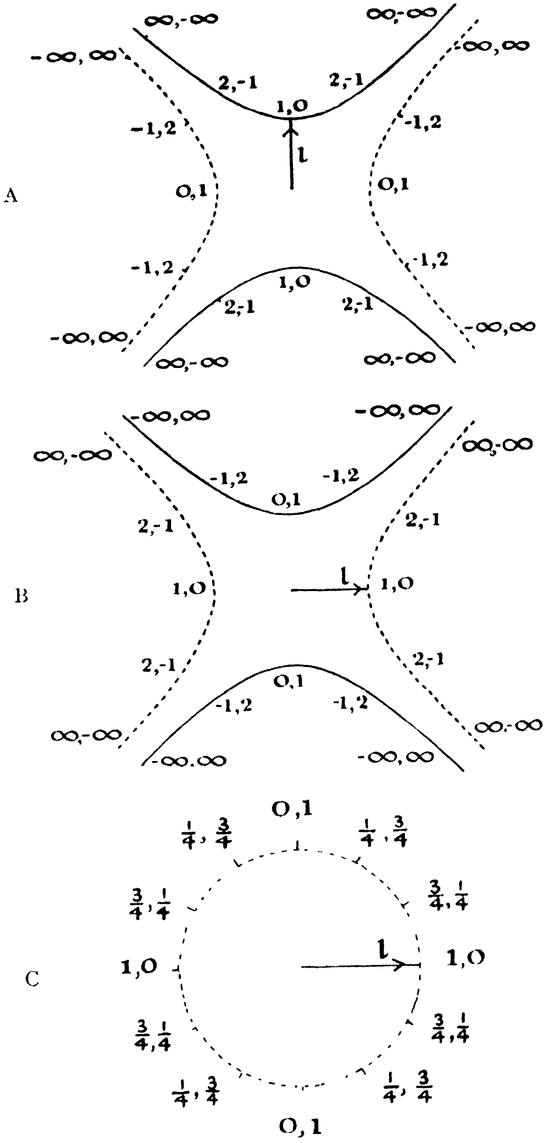
We adopt as one measure of the inclination $\nu_{OQ}/\nu_{\mathbf{m}}$ which we call the first inclination number and denote by $\theta_{\mathbf{lm}}$, and as another measure $\nu_{QM}/\nu_{\mathbf{m}}$ which we call the second inclination number and denote by $\phi_{\mathbf{lm}}$. We have thus

$$\theta_{\mathbf{lm}} = \varpi^2_{\mathbf{lm}}/\nu_{\mathbf{l}}\nu_{\mathbf{m}} \qquad \phi_{\mathbf{lm}} = \nu_{\mathbf{lm}}/\nu_{\mathbf{l}}\nu_{\mathbf{m}}$$

2. Each of the figures shows the intersection of contours + 1 and − 1 by a biplane, the full curve belonging to contour + 1 and the dotted curve to contour − 1. In each figure the vector **l** is given a fixed position; figure A is a mixed biplane with **l** timelike, figure B a mixed biplane with **l** spacelike, and figure C a spacelike biplane.

The head of vector **m** is to be thought of as traversing all the curves, and the values of $\theta_{\mathbf{lm}}$ $\phi_{\mathbf{lm}}$ are shown against the curves for a number of positions of **m**.

3. As the inclination numbers depend only on the directions of the vectors, there is no loss of generality in taking


∞,-∞
∞,-∞
-∞,∞
∞,∞
2,-1
2,-1
1,0
-1,2
-1,2
l
A
0,1
0,1
-1,2
-1,2
1,0
2,-1
2,-1
-∞,∞
-∞,∞
∞,-∞
∞,-∞
-∞,∞
-∞,∞
∞,-∞
∞,-∞
-1,2
-1,2
0,1
2,-1
2,-1
B
1,0
l
1,0
2,-1
2,-1
0,1
-1,2
-1,2
∞,-∞
∞,-∞
-∞,∞
-∞,∞
0,1
$\frac{1}{4}, \frac{3}{4}$
$\frac{1}{4}, \frac{3}{4}$
$\frac{3}{4}, \frac{1}{4}$
$\frac{3}{4}, \frac{1}{4}$
C
1,0
l
1,0
$\frac{3}{4}, \frac{1}{4}$
$\frac{3}{4}, \frac{1}{4}$
$\frac{1}{4}, \frac{3}{4}$
$\frac{1}{4}, \frac{3}{4}$
0,1

Figures for article 8,2.

unity as the size of both vectors. As any two conjugate diameters of the curves are at right angles in fourfold terminology and Euclidian perpendicularity does not enter in, there is no loss of generality in the use of the symmetrical form we have chosen for the position of vector **l**.

4. The results shown in the figures are summarized in the following table:

VALUE OF $\theta_{\mathbf{lm}}$	VALUE OF $\phi_{\mathbf{lm}}$	NATURE OF BIPLANE **lm**	NATURE OF VECTORS **l m**
$+\infty$	$-\infty$	mixed	both timelike or both spacelike
$+1$	0	spacelike	both spacelike
0	$+1$	mixed	one timelike and one spacelike
$-\infty$	$+\infty$		

5. We note that in Euclidian space, and in a spacelike biplane of the fourfold, it is possible to speak of the angle α between the vectors **l m** which has trigonometrical ratios such that

$$\cos^2 \alpha = \theta_{\mathbf{lm}} \quad \sin^2 \alpha = \phi_{\mathbf{lm}}$$

but that in other cases there is no real angle α that can satisfy these equations.

We note also that

$$\theta_{\mathbf{lm}} + \phi_{\mathbf{lm}} = 1$$

always, no matter whether the inclination can be given by an angle or not.

6. When one or both of the vectors **l m** approaches the tricone, the inclination numbers show peculiar properties.

When one of them **l** only approaches the tricone $\varpi_{\mathbf{lm}}$ approaches ∞ while $\nu_{\mathbf{l}}$ $\nu_{\mathbf{m}}$ remain finite, so that $\theta_{\mathbf{lm}}$ approaches ∞. When both approach the tricone the result depends on the order of approach. If $\mathbf{m}$ remains at a finite distance till $\mathbf{l}$ is on the tricone, we have seen that $\theta_{\mathbf{lm}} = \infty$. If $\mathbf{l}$ $\mathbf{m}$ first coincide and then approach the tricone, $\theta_{\mathbf{lm}}$ keeps the constant value unity.

We can also cause them to approach with any prescribed value for $\theta_{\mathbf{lm}}$. Suppose for example that the prescribed value is 2. OQ being the projection of $\mathbf{l}$ on $\mathbf{m} = \overrightarrow{OM}$, we

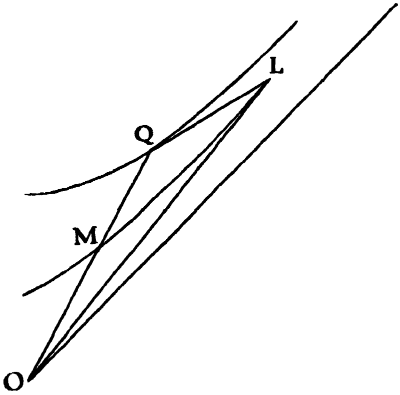


Figure for article 8,6.

obtain the result by keeping OQ equal to $2OM$ during the approach.

Intersection of a Biplane with the Contour Tricone

7. Using the notation of chapter 3 we take in the biplane $x = 0 \quad y = 0$ a point U with coordinates $0 \; 0 \; z \; t$. Through U we draw PUQ parallel to the y-axis to cut the tricone in $P \; Q$. If the coordinates of P are $0 \; y \; z \; t$, those of Q are $0 \; -y \; z \; t$, and y is given by

$$t^2 - y^2 - z^2 = 0$$

so that

$$\sigma_{UP} = \sigma_{UQ} = \sqrt{t^2 - z^2}$$

Also

$$\nu_{OU} = t^2 - z^2 \qquad \sigma_{OU} = \sqrt{t^2 - z^2}$$

Hence if we take *OY OU* as axes and draw the generators *OP OQ* in which the biplane *OPQ* cuts the tricone the generators will, in the Euclidian sense, be at right angles. And this is true however closely *OPQ* approaches the position of tangency to the tricone.

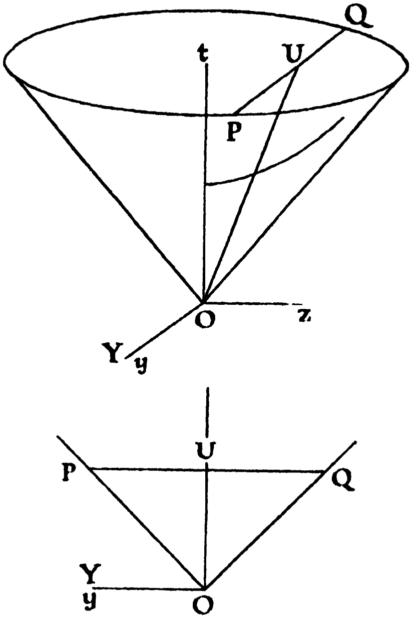


Figure for article 8,7.

When *OPQ* approaches the limiting position the two generators come in our picture very close together. This appearance is due to the inadequacy of our method of representation. In our picture the size shown for OU is a quasi-Euclidian size $\sqrt{t^2 + z^2}$, whereas the true size is the much smaller quantity $\sqrt{t^2 - z^2}$.

It is only in our Euclidian picture that the two generators are at right angles. In fourfold measure their first inclination number is ∞, while for fourfold perpendicularity it should be zero.

CHAPTER 9

THE ADDITION OF PARALLEL VECTOR QUANTITIES

1. To add two parallel line vectors we give one of them a parallel shift that brings it foot to head with the other. There are two cases according as the two vectors head the same way or different ways; in the figures $\overrightarrow{PR}$ is the sum of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

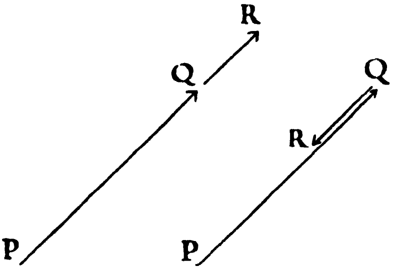


Figure for article 9,1.

2. Let **a** **b** be two vector parallelograms, in the same or in parallel biplanes, which we wish to add. Let **k** **l** be the sides of **a** and **r** **s** those of **b**.

Holding **r** still we slide the opposite side of **b** along its own line until **s** comes to the position **t**, parallel to **l**. Then holding **t** still we slide the opposite side of the parallelogram along its own line until **r** comes to the position **u** parallel to **k**. We reduce **t** to the size of **l** and increase **u** in the proportion in which **t** was reduced.

Taking account of aspect we write **a** = **kl**, and we put **b** equal to **ut** or to − **ut** according as **b** has the same aspect as **a** or the opposite aspect. In the former case we give **b** a parallel shift that brings Q, the foot of **u** and **t**, to K, the head of **k**; and we have the parallelogram $OXYL$ of figure B as the sum of **a** and **b**.

If **b** = − **ut** we bring the head of **u** to K, and so have as the sum of **a** and **b** the parallelogram $OXYL$ shown in

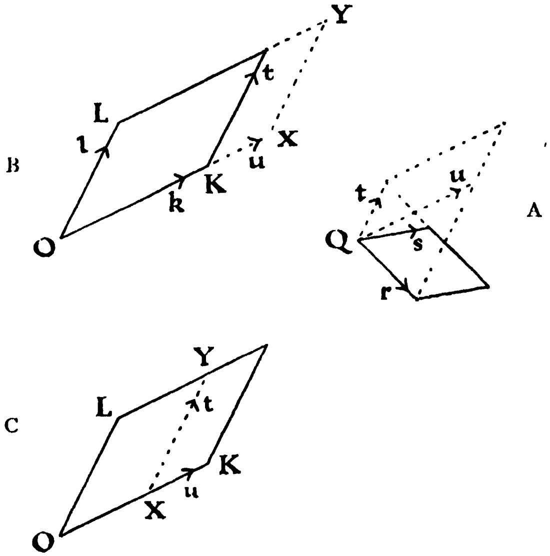


Figure for article 9,2.

figure C. We observe that if **a** and **b** have the same size and opposite aspects their sum is zero.

3. We could also begin by transforming **a** to a rectangle. That being supposed done and the rest of the construction carried out as before, the size of **a** is

$$\sigma_{\mathbf{a}} = \sigma_{\mathbf{k}}\sigma_{\mathbf{l}}$$

and the size of the sum of **a** and **b** is

$$\sigma_{OX}\sigma_{\mathbf{l}}$$

4. If an area vector of irregular outline has to be added in we first divide it up into rectangles by two series of parallel lines, lying in the biplane of the area vector, and the size of the original area is the sum of the sizes of the rectangles.

In particular if in a triangle OLM $\overrightarrow{OL} = \mathbf{l}$ and $\overrightarrow{OM} = \mathbf{m}$, the size of the triangle is $^1/_2\,\sigma_{\mathbf{lm}}$.

5. Let $\mathbf{v}$ $\mathbf{w}$ be two tripeds in the same or in parallel triplanes, $\mathbf{v}$ having edges $\mathbf{k}$ $\mathbf{l}$ $\mathbf{m}$ with their feet at O and $\mathbf{w}$ having edges $\mathbf{p}$ $\mathbf{q}$ $\mathbf{r}$ with their feet at Q.

Holding the base $\mathbf{pq}$ of $\mathbf{w}$ still we slide the opposite parallelogram face in its own plane to bring $\mathbf{r}$ to the position $\mathbf{u}$ parallel to $\mathbf{m}$. Then holding $\mathbf{pu}$ still we bring $\mathbf{q}$ to the position $\mathbf{t}$ parallel to $\mathbf{l}$ by a similar motion, and then holding $\mathbf{tu}$ still we bring $\mathbf{p}$ to the position $\mathbf{s}$ parallel to $\mathbf{k}$ in the same way.

We now multiply $\mathbf{u}$ by α to make it equal to $\mathbf{m}$, multiply $\mathbf{t}$ by β to make it equal to $\mathbf{l}$, and divide $\mathbf{s}$ by $\alpha\beta$, so that the three operations leave the size of $\mathbf{stu}$ unchanged.

Taking account of aspect we put $\mathbf{v} = \mathbf{klm}$, and $\mathbf{w}$ will be $\mathbf{stu}$ or $-\ \mathbf{stu}$ according as $\mathbf{w}$ looks the same way as $\mathbf{v}$ or the opposite way. If they look the same way we bring Q by a parallel shift to K the head of $\mathbf{k}$, so that $\mathbf{s}$ comes to the position $\overrightarrow{KZ}$ in figure B.

If $\mathbf{v}$ and $\mathbf{w}$ look opposite ways we use a parallel shift that brings $\mathbf{s}$ to the position $\overrightarrow{ZK}$ with its head at K, as in figure C. In particular if $\mathbf{v}$ and $\mathbf{w}$ have the same size but opposite aspects, their sum is zero.

In each case if we denote $\overrightarrow{OZ}$ by $\mathbf{z}$ we have the triped $\mathbf{zlm}$ as the sum of $\mathbf{v}$ and $\mathbf{w}$.

6. We could have begun by transforming the triped $\mathbf{v}$ to

a brick, that is to a rectangular triped. In such a case its size would be

$$\sigma_a = \sigma_k \sigma_l \sigma_m$$

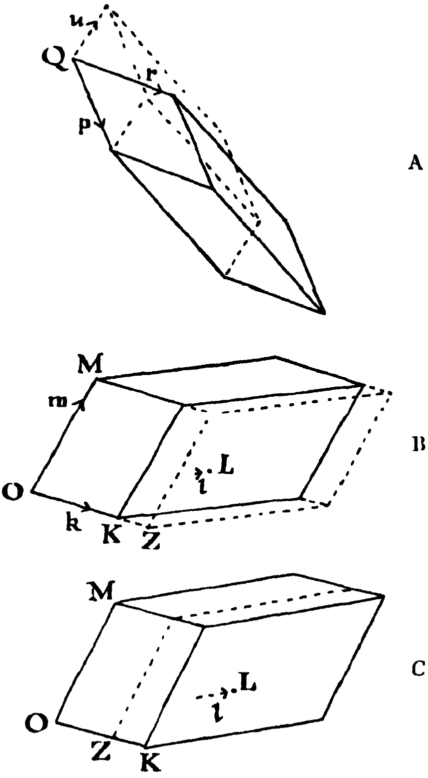


Figure for article 9,5.

and the size of the resulting brick

$$\sigma_z \sigma_l \sigma_m$$

7. To deal with an irregular piece of a triplane, that is a volume vector of irregular boundary, we divide it into bricks

by three sets of parallel biplanes lying in the triplane of the volume vector. The size of the volume vector is the sum of the sizes of the constituent bricks

In particular if $\mathbf{k} = \overrightarrow{OK}$ $\mathbf{l} = \overrightarrow{OL}$ $\mathbf{m} = \overrightarrow{OM}$ are three edges of a tetrahedron, its size is

$$\tfrac{1}{6}\,\sigma_{\mathbf{klm}}$$

one-sixth of the size of the triped with the same edges; and the triangular prism with base *OKL* and height *OM* has size

$$\tfrac{1}{2}\,\sigma_{\mathbf{klm}}$$

8. **Exercise.**—Consider the five-face of which $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ are four edges. Take as its base the tetrahedron that has $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ as edges. Let a triplane parallel to the tetrahedron move along $\mathbf{e}_4$. Compare the tetrahedron in which it cuts the five-face with the base, and by integration show that the size of the five-face is one quarter of the size of the superprism generated by the parallel shift of the base along $\mathbf{e}_4$.

9. **Exercise.**—Show that the five-face with $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$ for edges is $^1/_{24}$ of the tetraped $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$.

10. **Exercise.**—Show that the superprism generated by a parallel shift of a tetrahedron can be divided into four equal five-faces.

CHAPTER 10

ENCLOSURES

1. A ONEFOLD is a line, and two points in it separate the stretch between them from the rest of the onefold. Thus if we start from a point of the stretch and travel in either direction along the onefold we come up against one of the points. The stretch is thus enclosed by a barrier consisting of the two points.

If the line lies in a twofold and we are allowed free movement in the twofold, we can by stepping off the stretch come to any point of the line we please without meeting the barrier.

2. In a biplane (which is a twofold) the closed line made up of the four sides of a rectangle cuts off the enclosed piece from the rest of the twofold. And starting from a point in the rectangle and travelling parallel to either pair of sides or in any route compounded of these two directions we come up against the barrier and cannot reach the remainder of the twofold.

But if the biplane lies in a threefold in which we are free to move we can reach any point of the biplane without crossing the boundary; we have only to go along the third dimension perpendicular to the rectangle.

A closed line is thus a barrier in a twofold but not in a threefold.

3. In a triplane (which is a threefold) the six rectangle faces of a brick or rectangular triped separate a piece of the threefold from the rest. And starting from a point inside, in whatever direction we go, parallel to any of the three edge-directions, or in a direction compounded of these, we are always stopped by the barrier.

But if the triplane lies in the fourfold and we are free to move in the fourfold, we need only set off at right angles to the triplane in order to reach any desired point of the triplane without meeting the barrier.

Thus the bisurface of a brick is a barrier in a threefold but not in the fourfold.

4. Thus in every case a barrier in any manifold has one dimension less than the manifold. In a manifold with two dimensions more than the barrier it ceases to be a barrier.

CHAPTER II

GENERAL ADDITION OF VECTOR QUANTITIES

1. WE have had a case of the general addition of line vectors in the relation

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + x^4\mathbf{e}_4$$

between a vector line and its components. The principle may be stated in the form: *Any two routes from one point to another are vectorially equal*; or in the form: *The vector sum of a closed circuit, considered as composed of vector elements all heading the same way round the circuit, is zero.*

The definition that governs the addition of area vectors may be stated in the form: *The vector sum of a closed bisurface, considered as made up of elements of area all facing inwards, is zero*; or in the form: *Two bisurfaces, bounded by the same closed circuit, and looking the same way through the circuit, are vectorially equal.*

The definition of the addition of volume vectors may be stated in the form: *The vector sum of a closed trisurface, considered as made up of elementary volume vectors all facing inwards, is zero*; or in the form: *Two trisurfaces, bounded by the same closed bisurface, and looking the same way through it, are vectorially equal.*

2. Consider the circuit composed of a vector **l** and its four components. Fill in the circuit with a film or piece of bisurface which we will call *B*. Now give the film a parallel

shift along a vector **m**. It generates a piece of trisurface T bounded by a closed bisurface made up of 7 parts. Two of these are the film B in its initial and final positions, say B_1 the floor and B_2 the roof. The walls, the other 5 parts, are

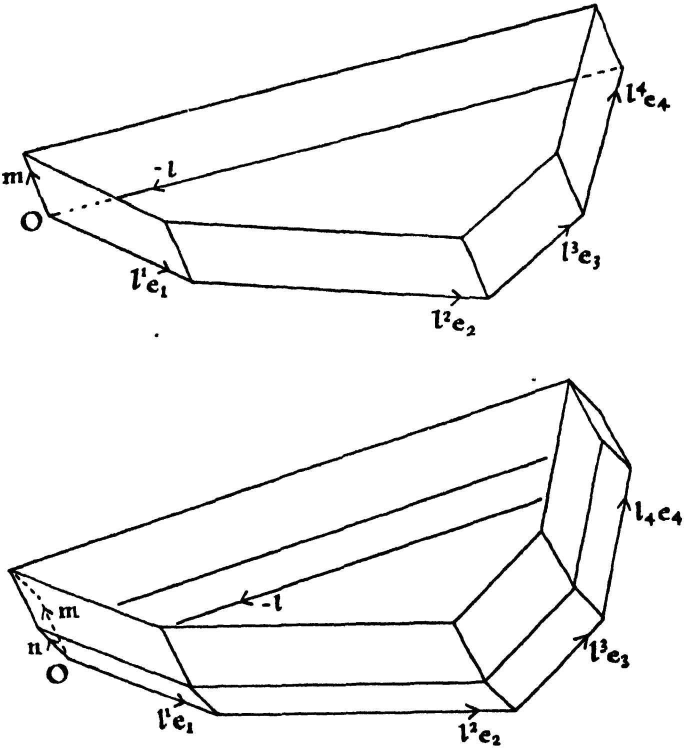


Figure for article 11,2.

the parallelograms generated by **l** and its four components.

The floor and the roof face one another, and corresponding elements are congruent and parallel, so that the vector sum of B_1 and B_2 is zero.

The elements of the circuit, all taken the same way round, are

$$l^1\mathbf{e}_1 \quad l^2\mathbf{e}_2 \quad l^3\mathbf{e}_3 \quad l^4\mathbf{e}_4 \quad -\mathbf{l}$$

so that the parallelograms they generate, taken with outward aspect, may be named

$$l^1\mathbf{e}_1\mathbf{m} \quad l^2\mathbf{e}_2\mathbf{m} \quad \mathbf{l}^3\mathbf{e}_3\mathbf{m} \quad l^4\mathbf{e}_4\mathbf{m} \quad -\mathbf{lm}$$

Since the sum of the whole boundary is zero, and also $B_1 + B_2 = 0$, it follows that the sum of these five quantities is zero; that is to say

$$\mathbf{lm} = l^1\mathbf{e}_1\mathbf{m} + l^2\mathbf{e}_2\mathbf{m} + l^3\mathbf{e}_3\mathbf{m} + l^4\mathbf{e}_4\mathbf{m}$$

3. **Exercise.**—Show that the film or bisurface which is to fill in the closed circuits in article 2 can be made up of a number of plane triangles.

4. We now give a parallel shift along a vector **n** to the piece of trisurface T. This generates a piece of fourfold F, whose boundary is made up of pieces of trisurface. Two of these are T_1 T_2 the initial and final positions of T; these we may call the (three-dimensional) floor and roof. The walls are the pieces of trisurface generated by the bisurface boundary of T, that is by the films B_1 B_2 and the five parallelograms.

The sum of the floor and roof is zero, since corresponding elements are congruent, parallel, and face different ways.

Take corresponding elements of B_1 and B_2. They are congruent, parallel, and face opposite ways. The two tripeds generated by the shift along **n** face opposite ways, and their sum is zero. Hence the sum of U_1 and U_2 the pieces of trisurface generated by B_1 B_2 is zero.

The five parallelograms generate the five tripeds

$$l^1\mathbf{e}_1\mathbf{mn} \quad l^2\mathbf{e}_2\mathbf{mn} \quad l^3\mathbf{e}_3\mathbf{mn} \quad l^4\mathbf{e}_4\mathbf{mn} \quad -\mathbf{lmn}$$

all facing inwards or all outwards. Since the sum of the complete trisurface boundary of F is zero, and the sum of T_1 and T_2 is zero, and the sum of U_1 U_2 is zero, it follows that the sum of the five tripeds is zero; that is to say

$$\mathbf{lmn} = l^1\mathbf{e}_1\mathbf{mn} + l^2\mathbf{e}_2\mathbf{mn} + l^3\mathbf{e}_3\mathbf{mn} + l^4\mathbf{e}_4\mathbf{mn}$$

5. We now give to the trisurface boundary of F a shift along a vector **p**. The boundary will take in as much new fourfold in front of it as it drops out behind. And we can measure these pieces of fourfold as the figures generated by T_1 T_2 U_1 U_2 and the five tripeds.

Two corresponding elements of T_1 T_2 are congruent, parallel, and face opposite ways, so that if the shift is in the direction of the positive face for one, it is away from the positive face for the other, and the sum of the two elements of fourfold is zero. Hence the sum of the pieces of fourfold generated by T_1 T_2 is zero.

And the same reasoning and result holds for U_1 U_2.

Hence the sum of the five tetrapeds generated by the five tripeds is zero. That is to say

$$\mathbf{lmnp} = l^1\mathbf{e}_1\mathbf{mnp} + l^2\mathbf{e}_2\mathbf{mnp} + l^3\mathbf{e}_3\mathbf{mnp} + l^4\mathbf{e}_4\mathbf{mnp}$$

6. Next we consider

$$l^1\mathbf{e}_1\mathbf{m} \qquad l^1\mathbf{e}_1\mathbf{mn} \qquad l^1\mathbf{e}_1\mathbf{mnp}$$

and by similar procedure starting from the circuit formed by **m** and its components, we obtain expressions for these quantities in terms of such quantities as

$$l^1\mathbf{e}_1m^2\mathbf{e}_2 \qquad l^1\mathbf{e}_1m^2\mathbf{e}_2\mathbf{n} \qquad l^1\mathbf{e}_1m^2\mathbf{e}_2\mathbf{np}$$

7. We proceed similarly from the corresponding circuits for **n** and **p** and ultimately arrive at expressions in terms of

the components of all four vectors, and thus prove that in the expressions

$$\mathbf{lm} \qquad \mathbf{lmn} \qquad \mathbf{lmnp}$$

it is legitimate to replace each vector by the sum of its components and multiply out as in elementary algebra, except that we maintain the order of the vectors $\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_4$.

8. We may also use the derived lattice $\mathbf{e}^{1234}$ in place of the original lattice $\mathbf{e}_{1234}$ and so obtain corresponding expressions.

9. **Exercise.**—Evaluate $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^3\mathbf{e}^4$ and hence show that $g_{1212} = gg^{3434}$.

10. **Exercise.**—Evaluate $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^\alpha\mathbf{e}^\beta$ and show that

$$g_{12\gamma\delta} = gg^{34\alpha\beta}$$

where $\alpha \quad \beta \quad \gamma \quad \delta$ are an even permutation from $1 \quad 2 \quad 3 \quad 4$.

The Pointer of a Triped

11. For the closer study of the aspect of a triped $\mathbf{v}$ $\mathbf{klm}$ we shall associate the normal from the triplane on one side with the aspect of $\mathbf{klm} = \mathbf{lmk} = \mathbf{mkl}$, and the normal on the other side with the aspect of $\mathbf{mlk} = \mathbf{lkm}$ $\mathbf{kml}$.

Thus in the $\mathbf{e}$ lattice the triped $\mathbf{klm}$ can be written* $^1/_6\,\overline{klm}^{\alpha\beta\gamma}\mathbf{e}_{\alpha\beta\gamma}$ and the normal which we associate with it is†

$$\mathbf{p} = (\overline{klm}^{432}\mathbf{e}^1 + \overline{klm}^{341}\mathbf{e}^2 + \overline{klm}^{214}\mathbf{e}^3 + \overline{klm}^{123}\mathbf{e}^4)\sqrt{-g}$$

---

[*] By article 7 we obtain the value of $\mathbf{klm}$ by multiplying out $(k^1\mathbf{e}_1 + k^2\mathbf{e}_2 + k^3\mathbf{e}_3 + k^4\mathbf{e}_4)(l^1\mathbf{e}_1 + l^2\mathbf{e}_2 + l^3\mathbf{e}_3 + l^4\mathbf{e}_4)(m^1\mathbf{e}_1 + m^2\mathbf{e}_2 + m^3\mathbf{e}_3 + m^4\mathbf{e}_4)$. Taking account of the fact that $\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2$ we find for the coefficient of $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

$$\left.\begin{matrix} k^1 & l^1 & m^1 \\ k^2 & l^2 & m^2 \\ k^3 & l^3 & m^3 \end{matrix}\right|$$

which we write shortly as $\overline{klm}^{123}$, and so for the other three terms. The factor $^1/_6$ enters into the expression because we can make $\alpha \quad \beta \quad \gamma$ equal to $1 \quad 2 \quad 3$ in six ways.

[†] The perpendicularity of $\mathbf{p}$ to $\mathbf{klm}$ is shown by calculating $\mathbf{p}\cdot\mathbf{k} \quad \mathbf{p}\cdot\mathbf{l} \quad \mathbf{p}\cdot\mathbf{m}$.

This normal **p** and its triped **klm** are unaltered by cyclical change of **k** **l** **m**, while the triped $-\mathbf{v} \equiv \mathbf{mlk}$ and its associated normal

$$-\mathbf{p} \equiv (\overline{mlk}^{432}\mathbf{e}^1 + \overline{mlk}^{341}\mathbf{e}^2 + \overline{mlk}^{214}\mathbf{e}^3 + \overline{mlk}^{123}\mathbf{e}^4)\sqrt{-g}$$

are also unaltered by cyclical interchange of **k** **l** **m**.

The normal associated with a triped we also call the pointer of the triped or of the triplane.

12. The pointer of **klm** may also be written as

$$\mathbf{p} = \overline{klm\mathbf{e}}^{1234}\sqrt{-g} \equiv \begin{vmatrix} k^1 & k^2 & k^3 & k^4 \\ l^1 & l^2 & l^3 & l^4 \\ m^1 & m^2 & m^3 & m^4 \\ \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 \end{vmatrix} \sqrt{-g}$$

If we express the original coordinates in terms of the derived coordinates this expression becomes

$$\begin{vmatrix} g^{1\alpha} & g^{2\alpha} & g^{3\alpha} & g^{4\alpha} \\ g^{1\beta} & g^{2\beta} & g^{3\beta} & g^{4\beta} \\ g^{1\gamma} & g^{2\gamma} & g^{3\gamma} & g^{4\gamma} \\ g^{1\delta} & g^{2\delta} & g^{3\delta} & g^{4\delta} \end{vmatrix} k_\alpha l_\beta m_\gamma \mathbf{e}_\delta \sqrt{-g}$$

The determinant has the value $1/g$ when α β γ δ is an even permutation, $-1/g$ for an odd permutation, and zero if any two of α β γ δ are equal. So that for **p** we have

$$\mathbf{p} = \frac{1}{g}\overline{klm\mathbf{e}}_{1234}\sqrt{-g} = -\overline{klm\mathbf{e}}_{1234}/\sqrt{-g}$$

or written out in full

$$\mathbf{p} = -(\overline{klm}_{432}\mathbf{e}_1 + \overline{klm}_{341}\mathbf{e}_2 + \overline{klm}_{214}\mathbf{e}_3 + \overline{klm}_{123}\mathbf{e}_4)/\sqrt{-g},$$

13. The expressions for **p** in terms of $\mathbf{e}^\alpha$ and of $\mathbf{e}_\alpha$ give*

$$\mathbf{p} \cdot \mathbf{p} = -\mathbf{v} \cdot \mathbf{v}$$

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* This involves a property that is proved later (in article 12,8) namely:

$$\mathbf{v} \cdot \mathbf{v} = \overline{klm}^{432}\overline{klm}_{432} + \overline{klm}^{341}\overline{klm}_{341} + \overline{klm}^{214}\overline{klm}_{214} + \overline{klm}^{123}\overline{klm}_{123}$$

so that a spacelike triped has a timelike pointer, and a mixed triped has a spacelike pointer. Of the two size-numbers one is positive and one is negative, but the sizes of triped and pointer are the same.

14. By putting $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ for **k** **l** **m** we find $\mathbf{e}^4\sqrt{-g}$ for the pointer of $\mathbf{e}_{123}$, and by putting $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ for **k** **l** **m** we find $-\mathbf{e}_4/\sqrt{-g}$ for the pointer of $\mathbf{e}^{123}$. And in general tripeds $\mathbf{e}_{\alpha\beta\gamma}$ $\mathbf{e}^{\alpha\beta\gamma}$ have pointers $\mathbf{e}^\delta\sqrt{-g}$ and $-\mathbf{e}_\delta/\sqrt{-g}$ where α β γ δ are an even permutation.

15. We now use a supercubical lattice with $\mathbf{e}_4$ as time vector, that is to say one in which g_{44} is 1, g_{11} g_{22} g_{33} have the value -1, and the cross constants vanish. In this lattice

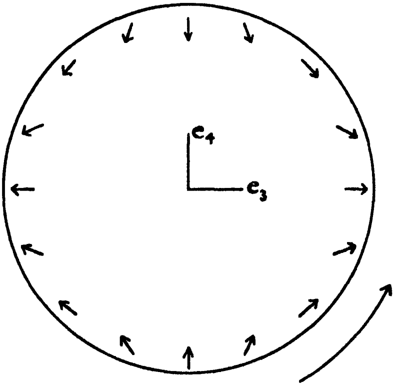


Figure A for article 11,15.

$$\mathbf{e}^4 = \mathbf{e}_4 \quad \mathbf{e}^1 = -\mathbf{e}_1 \quad \mathbf{e}^2 = -\mathbf{e}_2 \quad \mathbf{e}^3 = -\mathbf{e}_3 \quad g = -1$$

Figure A shows a circuit in the biplane $\mathbf{e}_3\mathbf{e}_4$. We suppose it to be described in the direction of the curved arrow, and we suppose the vector **k** to be an element of the circuit. Thus **k** has the form $k^3\mathbf{e}_3 + k^4\mathbf{e}_4$.

Consider the triped $\mathbf{e}_1\mathbf{e}_2\mathbf{k}$, that is to say the triped of which three edges are **k** and parallels to $\mathbf{e}_1$ $\mathbf{e}_2$ drawn through the foot of **k**. If we take such a triped for every element **k** of the circuit, they form a trisurface which we may call cylindrical. The figure is the intersection of this trisurface by $\mathbf{e}_3\mathbf{e}_4$. The trisurface surrounds and partially encloses a piece of fourfold.

16. The triped $\mathbf{e}_1\mathbf{e}_2\mathbf{k}$ is $k^3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - k^4\mathbf{e}_2\mathbf{e}_1\mathbf{e}_4$ and its pointer

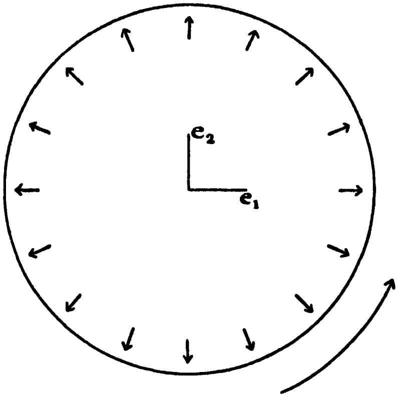


Figure B for article 11,16.

is $k^3\mathbf{e}^4 - k^4\mathbf{e}^3$ or $k^3\mathbf{e}_4 + k^4\mathbf{e}_3$. The figure shows the direction of the pointer at various points of the circuit.

Figure B is the corresponding figure for the intersection of $\mathbf{e}_3\mathbf{e}_4(k^1\mathbf{e}_1 + k^2\mathbf{e}_2)$ by the biplane $\mathbf{e}_1\mathbf{e}_2$.

We observe that in these cases all timelike pointers head inwards and all spacelike pointers outwards, and that for any trisurface enclosing a piece of fourfold if we pass continuously from element to element of the trisurface we shal find spacelike and timelike pointers heading different ways.

17. Thus in the figures we pass from the spacelike piece of trisurface $\mathbf{e}_1\mathbf{e}_2\mathbf{k}$ with a timelike pointer **p** heading inwards to the piece $\mathbf{e}_1\mathbf{e}_2\mathbf{l}$. If as in figure A this piece is also spacelike its pointer **q** is timelike and heads inwards, but if as in figure B $\mathbf{e}_1\mathbf{e}_2\mathbf{l}$ is of mixed character and has a spacelike pointer **q**, then **q** heads outwards.

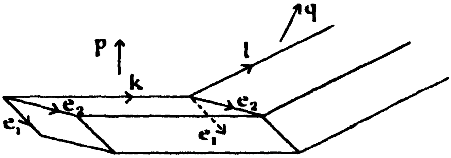


Figure A for article 11,17.

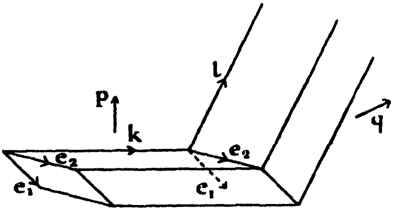


Figure B for article 11,17.

18. We have hitherto spoken of the elements of the boundary of an enclosure as facing inwards on the enclosed region. It appears now that we must correct that form of speech. If the timelike pointers head inwards the spacelike ones head outwards. To say this every time however would be tedious, and we shall continue to speak of boundaries "facing inwards" or as being of "inward aspect," with the convention that we thereby mean that timelike pointers head inwards and others outwards.

19. We consider the passage from one prime triped to another at the origin, say from $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ to $\mathbf{e}_4\mathbf{e}_2\mathbf{e}_3$. In this case it is $-\mathbf{e}_1$ which passes continuously into $\mathbf{e}_4$. To the former piece of trisurface we give the name $\mathbf{e}_3\mathbf{e}_2(-\mathbf{e}_1)$. This is the same as $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ and has timelike pointer $\mathbf{e}^4$ or $\mathbf{e}_4$, and faces inwards on the tetraped $\mathbf{e}_{1234}$.

$\mathbf{e}_3\mathbf{e}_2(-\mathbf{e}_1)$ passes into $\mathbf{e}_3\mathbf{e}_2\mathbf{e}_4$ or $\mathbf{e}_4\mathbf{e}_3\mathbf{e}_2$ which has pointer $\mathbf{e}^1$ or $-\mathbf{e}_1$, a spacelike pointer heading outwards as it ought.

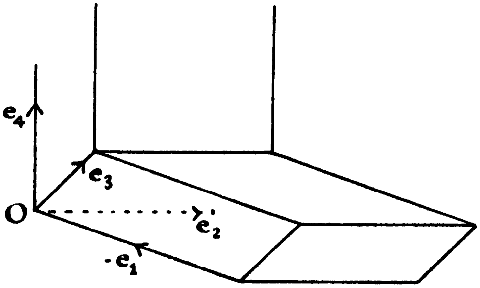


Figure for article 11,19.

20. In the same way $\mathbf{e}_1\mathbf{e}_3(-\mathbf{e}_2)$ passes into $\mathbf{e}_1\mathbf{e}_3\mathbf{e}_4$ with outward pointer $\mathbf{e}^2$ or $-\mathbf{e}_2$, and $\mathbf{e}_2\mathbf{e}_1(-\mathbf{e}_3)$ passes into $\mathbf{e}_2\mathbf{e}_1\mathbf{e}_4$ with outward pointer $\mathbf{e}^3$ or $-\mathbf{e}_3$. Thus appropriate names for the prime tripeds considered as boundaries of the tetraped $\mathbf{e}_{1234}$ are

$$\mathbf{e}_{123} \quad \mathbf{e}_{432} \quad \mathbf{e}_{341} \quad \mathbf{e}_{214}$$

and the appropriate aspects for the four parallel boundaries of the tetraped are given by

$$-\mathbf{e}_{123} \quad -\mathbf{e}_{432} \quad -\mathbf{e}_{341} \quad -\mathbf{e}_{214}$$

21. In the preceding articles we have used a supercubical lattice in order to bring out the facts about the aspect of the bounding trisurface. The principle of continuous passage

from one piece of bounding trisurface to the next is independent of that lattice, and the names given in article 20 to the eight trisurface boundaries of the prime tetraped are valid in all lattices.

22. In any **e**-lattice let four vectors **k** **k** + **p** **k** + **q** **k** + **r** be drawn parallel to $\mathbf{e}_4$ from the origin and from the heads of $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$. Let vectors **x** **y** **z** join the heads of these four vectors in such a way that **x** **y** **z** project into $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$. We consider the piece of fourfold generated by the triped **xyz** in its descent to coincidence with $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, and we discuss its trisurface boundary.

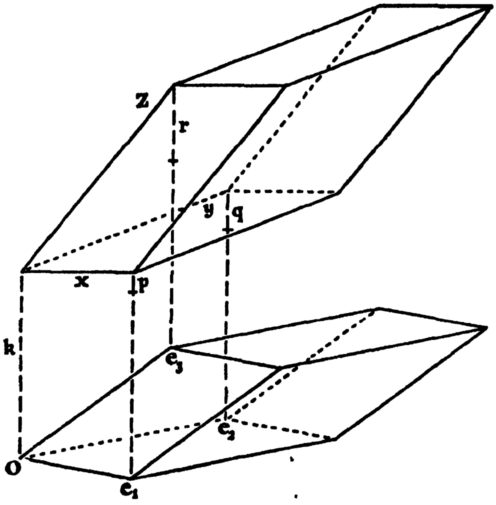


Figure for article 11,22.

23. One trisurface wall T is the figure generated by the descent of the parallelogram **yz** to coincidence with $\mathbf{e}_2\mathbf{e}_3$. This wall has the orientation and aspect of $\mathbf{e}_3\mathbf{e}_2\mathbf{e}_4$ but differs

in size. As to size $\mathbf{e}_3$ $\mathbf{e}_2$ give its extent in two directions while the edges in the third direction parallel to $\mathbf{e}_4$ are

$$\mathbf{k} \qquad \mathbf{k}+\mathbf{q} \qquad \mathbf{k}+\mathbf{r} \qquad \mathbf{k}+\mathbf{q}+\mathbf{r}$$

The average of these is $\mathbf{k} + (\mathbf{q} + \mathbf{r})/2$ and the wall *T* is equal to the triped

$$\mathbf{e}_3\mathbf{e}_2\{\mathbf{k} + (\mathbf{q} + \mathbf{r})/2\}$$

in size orientation and aspect.

24. The wall *U* parallel to *T* is generated by the descent to the triplane $\mathbf{e}_{123}$ of the parallelogram face of **xyz** which is opposite to **yz**. The projectors of the four corners are therefore greater by **p** than those of the corners of **yz**; they are

$$\mathbf{k}+\mathbf{p} \qquad \mathbf{k}+\mathbf{q}+\mathbf{p} \qquad \mathbf{k}+\mathbf{r}+\mathbf{p} \qquad \mathbf{k}+\mathbf{q}+\mathbf{r}+\mathbf{p}$$

and the wall *U* whose aspect is $-\mathbf{e}_3\mathbf{e}_2\mathbf{e}_4$ is equal in size orientation and aspect to the triped

$$-\mathbf{e}_3\mathbf{e}_2\{\mathbf{k} + (\mathbf{q} + \mathbf{r})/2 + \mathbf{p}\}$$

25. Hence the vector sum of the walls *T* *U* is

$$-\mathbf{e}_3\mathbf{e}_2\mathbf{p}$$

and we find a similar expression from each of the other two pairs of parallel walls. In addition we have $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ which we may call the floor, and $-$ **xyz** the roof. Equation of the whole boundary to zero gives

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - \mathbf{xyz} - \mathbf{e}_3\mathbf{e}_2\mathbf{p} - \mathbf{e}_1\mathbf{e}_3\mathbf{q} - \mathbf{e}_2\mathbf{e}_1\mathbf{r} = 0$$

26. Since two tripeds on the same base and between the same parallel biplanes are equal, we may write **zyp** for $\mathbf{e}_3\mathbf{e}_2\mathbf{p}$. And we can identify any four vectors we please with the prime vectors $\mathbf{e}_\alpha$. So we may state the result as follows :—

Let **xyz** be any triped, let the projections of **x** **y** **z** on a vector **n** be **p** **q** **r**, and let **stu** be the projection of **xyz** on a triplane normal to **n**; then

$$\mathbf{stu} = \mathbf{xyz} - \mathbf{pyz} - \mathbf{xqz} - \mathbf{xyr}$$

27. By the definition in article 1 two pieces of trisurface T_1 T_2 which are bounded by the same closed bisurface B and face the same way are vectorially equal.

We take the closed bisurface B as made up of the six parallelogram boundaries of the triped **klu** shown in the figure. As T_1, one piece of trisurface bounded by it, we take the triped **klu** itself.

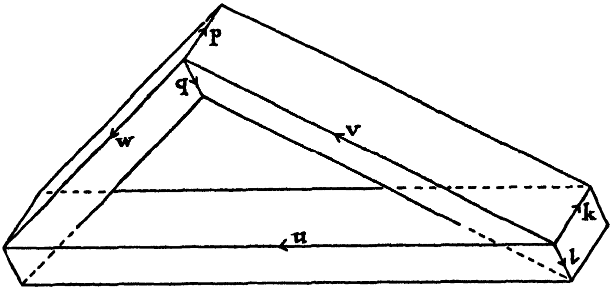


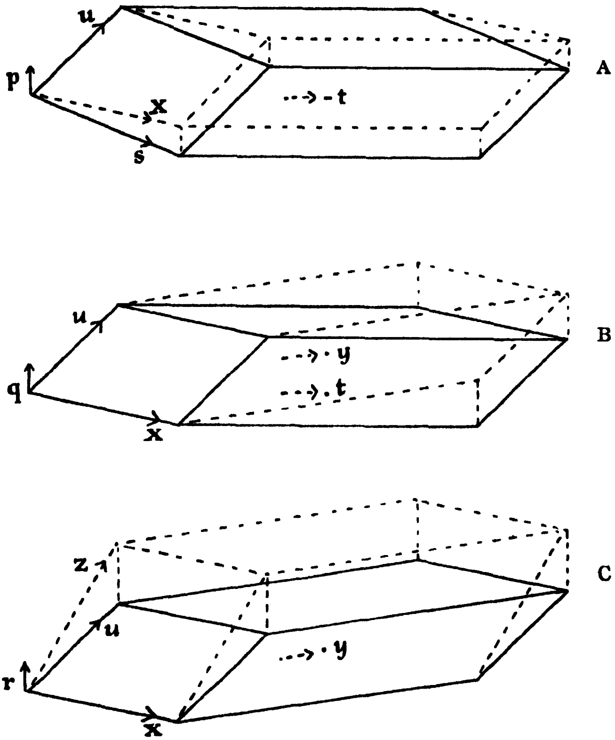
Figure for article 11,27.

As the other piece T_2 we take a figure made up of six parts each part standing on one of the six parallelograms of B. One of these is the triangular slab P_1 which has as base the triangle enclosed by **u** **v** **w** and as height **k**, and which stands on the parallelogram **ku**. Another P_2 is a triangular slab congruent to P_1, it is P_1 shifted along vector **l**, and stands on the parallelogram face opposite to **ku**.

A third triangular slab Q_1 has edges **u** **v** **w** **l** and stands on the parallelogram **lu**; and a fourth Q_2 congruent to it stands on the parallelogram opposite to **lu**.

We complete the trisurface T_2 by the two tripeds **klv** **pqw** which stand on the remaining two parallelograms and join up the four triangular slabs.

28. The two slabs P_1 P_2 are congruent and have the same orientation, and considered as portions of T_2 they have opposite aspects. So that their sum is zero.



Figures for article 11,29

In the same way Q_1 Q_2 have a zero sum.

For the expression of T_2 considered as facing the same way as T_1 we have therefore the expression **klv** + **pqw** or **klv** + **klw**. Hence by equating T_2 and T_1

$$\mathbf{klu} = \mathbf{klv} + \mathbf{klw}$$

which equation expresses one triped in terms of two component tripeds.

This is of course a particular case of article 4, and could also be derived from the equality of two tripeds on the same base and between the same parallel biplanes.

29. Again let us start with the triped **stu** in figure A. We take a vector **p** parallel to a given vector **n**, and take **x** equal to **s** + **p**. Article 28 tells us that triped **xtu** is equal to the sum of **stu** and **ptu**.

We now in figure B take the triped **xtu**, take **q** parallel to **n**, and make **y** = **t** + **q**. Then triped **xyu** is the sum of **xtu** and **xqu**.

Now in figure C we take the triped **xyu**, and add to **u** a a vector **r** parallel to **n**, giving **z** = **u** + **r**. Then triped **xyz** is the sum of **xyu** and **xyr**.

Thus we have

$$\mathbf{xyz} = \mathbf{xyr} + \mathbf{xqu} + \mathbf{ptu} + \mathbf{stu}$$

We note that, tripeds on the same base and between the same parallels being equal, **xqu** is equal to **xqz** and **ptu** is equa to **pyz**, and that **stu** is the projection of **xyz** parallel to **n** Thus our result is equivalent to that of article 26.

CHAPTER 12

EXPRESSIONS FOR CERTAIN OUTER AND INNER PRODUCTS

1. CONSIDER the vector parallelogram* $\mathbf{lm} \equiv l^\alpha \mathbf{e}_\alpha m^\beta \mathbf{e}_\beta$. We expand this expression in accordance with article 11,7, remembering that $\mathbf{e}_\beta \mathbf{e}_\alpha = -\mathbf{e}_\alpha \mathbf{e}_\beta$, and obtain

$$\begin{aligned}\mathbf{lm} = {} & (l^1 m^2 - l^2 m^1)\mathbf{e}_{12} + (l^1 m^3 - l^3 m^1)\mathbf{e}_{13} + (l^1 m^4 - l^4 m^1)\mathbf{e}_{14} \\ & + (l^2 m^3 - l^3 m^2)\mathbf{e}_{23} + (l^3 m^4 - l^4 m^3)\mathbf{e}_{34} + (l^4 m^2 - l^2 m^4)\mathbf{e}_{42}\end{aligned}$$

We denote $\mathbf{lm}$ by $\mathbf{a}$, $l^\alpha m^\beta - l^\beta m^\alpha$ by $\overline{lm}^{\alpha\beta}$ and by $a^{\alpha\beta}$, and $l_\alpha m_\beta - l_\beta m_\alpha$ by $\overline{lm}_{\alpha\beta}$ and by $a_{\alpha\beta}$. Our expression becomes

$$\mathbf{a} = \mathbf{lm} = {}^1/_2\, a^{\alpha\beta}\mathbf{e}_{\alpha\beta} = {}^1/_2\, \overline{lm}^{\alpha\beta}\mathbf{e}_{\alpha\beta}$$

the factor ${}^1/_2$ entering because, for example, $a^{12}\mathbf{e}_{12}$ is given by $\alpha = 2 \quad \beta = 1$ as well as by $\alpha = 1 \quad \beta = 2$.

The expression similarly obtained from the derived lattice $\mathbf{e}^{1234}$ is $$\mathbf{a} = \mathbf{lm} = {}^1/_2\, a_{\alpha\beta}\mathbf{e}^{\alpha\beta} = {}^1/_2\, \overline{lm}_{\alpha\beta}\mathbf{e}^{\alpha\beta}$$

2. More concisely we could obtain the result of article 1 as follows: $\mathbf{a}$ has the value $l^\alpha \mathbf{e}_\alpha m^\beta \mathbf{e}_\beta$, or by interchanging α and β $l^\beta \mathbf{e}_\beta m^\alpha \mathbf{e}_\alpha$. Adding these we have

$$\begin{aligned}2\mathbf{a} &= l^\alpha m^\beta \mathbf{e}_{\alpha\beta} - l^\beta m^\alpha \mathbf{e}_{\alpha\beta} \\ &= (l^\alpha m^\beta - l^\beta m^\alpha)\mathbf{e}_{\alpha\beta}\end{aligned}$$

or $$\mathbf{a} = {}^1/_2\, \overline{lm}^{\alpha\beta}\mathbf{e}_{\alpha\beta} = {}^1/_2\, a^{\alpha\beta}\mathbf{e}_{\alpha\beta}$$

---

* The vector parallelogram $\mathbf{lm}$ is called the outer product of the line vectors $\mathbf{l}$ $\mathbf{m}$, the vector triped $\mathbf{lmn}$ is the outer product of $\mathbf{l}$ $\mathbf{m}$ $\mathbf{n}$, and the tetraped $\mathbf{lmnp}$ the outer product of $\mathbf{l}$ $\mathbf{m}$ $\mathbf{n}$ $\mathbf{p}$.

3. We have thus found twelve parallelograms lying parallel to the six prime biplanes $\mathbf{e}_{\alpha\beta}$ which are together vectorially equal to the original parallelogram **a**. Taking them in pairs we call them the six components

$$a^{13}\mathbf{e}_{13} \quad a^{12}\mathbf{e}_{12} \quad a^{14}\mathbf{e}_{14} \quad a^{23}\mathbf{e}_{23} \quad a^{34}\mathbf{e}_{34} \quad a^{42}\mathbf{e}_{42}$$

of **a**.

The procedure in chapter 11 was to replace the original parallelogram by four others together with two films that cancelled one another, and then to replace each of the four by three others (or four others one of which was zero) together with a pair of mutually cancelling films. Thus we finished with a bisurface having the same boundary line as **a** and consisting of the 12 parallelograms together with a number of mutually cancelling films.

4. Although it took eight components to fix the two vectors **l** **m** there are only six components of the area vector $\mathbf{a} \equiv \mathbf{lm}$, and as we shall show any one of these six can be expressed in terms of the other five

The determinant $\begin{vmatrix} l^\alpha & l^\beta & l^\gamma & l^\delta \\ m^\alpha & m^\beta & m^\gamma & m^\delta \\ l^\alpha & l^\beta & l^\gamma & l^\delta \\ m^\alpha & m^\beta & m^\gamma & m^\delta \end{vmatrix}$

vanishes, and by appropriate expansion of it we find

$$\overline{lm}^{\alpha\beta}\overline{lm}^{\gamma\delta} + \overline{lm}^{\beta\gamma}\overline{lm}^{\alpha\delta} + \overline{lm}^{\gamma\alpha}\overline{lm}^{\beta\delta} = 0$$

or

$$\begin{vmatrix} a^{\alpha\gamma} & a^{\alpha\delta} \\ a^{\beta\gamma} & a^{\beta\delta} \end{vmatrix} = a^{\alpha\beta}a^{\gamma\delta}$$

This is true for all values of α β γ δ, and when they are all different it is

$$a^{12}a^{34} + a^{23}a^{14} + a^{31}a^{24} = 0$$

5. **Exercise.**—Having given six quantities $a^{\rho\sigma}$ for which $a^{\sigma\rho} = -a^{\rho\sigma}$ and $a^{12}a^{34} + a^{23}a^{14} + a^{31}a^{24} = 0$, show that two vectors $\mathbf{l}$ $\mathbf{m}$ can be found such that $\overline{lm}^{\rho\sigma} = a^{\rho\sigma}$, that is to say an area vector exists of which $a^{\rho\sigma}$ are the components.

6. For the size-number of **a** we had in article 5,7 the expression

$$\nu_{\mathbf{lm}} = \begin{vmatrix} \mathbf{l}\cdot\mathbf{l} & \mathbf{l}\cdot\mathbf{m} \\ \mathbf{m}\cdot\mathbf{l} & \mathbf{m}\cdot\mathbf{m} \end{vmatrix}$$

We had also

$$\mathbf{l}\cdot\mathbf{l} = l^{\alpha}l_{\alpha} \qquad \mathbf{l}\cdot\mathbf{m} = l^{\alpha}m_{\alpha}$$

so that

$$\nu_{\mathbf{lm}} = \begin{vmatrix} l^{\alpha}l_{\alpha} & l^{\alpha}m_{\alpha} \\ m^{\beta}l_{\beta} & m^{\beta}m_{\beta} \end{vmatrix}$$

the affixes α β being used in such a way that each occurs twice and not four times in any term.

We have

$$\nu_{\mathbf{lm}} = l^{\alpha}m^{\beta}(l_{\alpha}m_{\beta} - l_{\beta}m_{\alpha})$$

and by interchanging α and β

$$\nu_{\mathbf{lm}} = -\,l^{\beta}m^{\alpha}(l_{\alpha}m_{\beta} - l_{\beta}m_{\alpha})$$

and by adding these two equations

$$\nu_{\mathbf{lm}} = {}^{1}/_{2}\,(l^{\alpha}m^{\beta} - l^{\beta}m^{\alpha})(l_{\alpha}m_{\beta} - l_{\beta}m_{\alpha})$$

or

$$\nu_{\mathbf{a}} = {}^{1}/_{2}\,a^{\alpha\beta}a_{\alpha\beta}.$$

The notation $\mathbf{a}\cdot\mathbf{a}$ is also used for the size-number of **a**.

7. We denote the vector triped **lmn** by **v**, the determinant

$$\begin{vmatrix} l^{\alpha} & l^{\beta} & l^{\gamma} \\ m^{\alpha} & m^{\beta} & m^{\gamma} \\ n^{\alpha} & n^{\beta} & n^{\gamma} \end{vmatrix}$$

by $\overline{lmn}^{\alpha\beta\gamma}$ and by $v^{\alpha\beta\gamma}$, with the same notation for lower affixes. We have

$$\mathbf{v} = l^{\alpha}\mathbf{e}_{\alpha}m^{\beta}\mathbf{e}_{\beta}n^{\gamma}\mathbf{e}_{\gamma}$$

where the right side is the product of three factors each of which has four terms, and by article 11,7 multiplication is to proceed as in elementary algebra except that

$$\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma = \mathbf{e}_\beta\mathbf{e}_\gamma\mathbf{e}_\alpha = \mathbf{e}_\gamma\mathbf{e}_\alpha\mathbf{e}_\beta = -\,\mathbf{e}_\gamma\mathbf{e}_\beta\mathbf{e}_\alpha = -\,\mathbf{e}_\alpha\mathbf{e}_\gamma\mathbf{e}_\beta = -\,\mathbf{e}_\beta\mathbf{e}_\alpha\mathbf{e}_\gamma$$

and that $\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma$ vanishes if two affixes are equal. We can therefore write

$$\mathbf{v} = l^\beta\mathbf{e}_\beta m^\gamma\mathbf{e}_\gamma n^\alpha\mathbf{e}_\alpha = l^\beta m^\gamma n^\alpha\mathbf{e}_\beta\mathbf{e}_\gamma\mathbf{e}_\alpha = l^\beta m^\gamma n^\alpha\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma$$

and so for each of the six forms obtained by permuting $\alpha \quad \beta \quad \gamma$. Then by addition of the six

$$6\mathbf{v} = \begin{vmatrix} l^\alpha & l^\beta & l^\gamma \\ m^\alpha & m^\beta & m^\gamma \\ n^\alpha & n^\beta & n^\gamma \end{vmatrix} \mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma$$

or
$$\mathbf{v} = {}^1/_6\, \overline{lmn}^{\alpha\beta\gamma}\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma = {}^1/_6\, v^{\alpha\beta\gamma}\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma$$

or if we write $\mathbf{e}_{\alpha\beta\gamma}$ to denote $\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma$

$$v = {}^1/_6\, v^{\alpha\beta\gamma}\mathbf{e}_{\alpha\beta\gamma}$$

The corresponding expression starting from lattice $\mathbf{e}^{1234}$ is

$$\mathbf{v} = {}^1/_6\, v_{\alpha\beta\gamma}\mathbf{e}^{\alpha\beta\gamma}$$

8. For the size-number $\nu_\mathbf{v}$ which we also denote by $\mathbf{v}\cdot\mathbf{v}$ we use the expression of article 6,8 and have

$$\nu_\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = \begin{vmatrix} \mathbf{l}\cdot\mathbf{l} & \mathbf{l}\cdot\mathbf{m} & \mathbf{l}\cdot\mathbf{n} \\ \mathbf{m}\cdot\mathbf{l} & \mathbf{m}\cdot\mathbf{m} & \mathbf{m}\cdot\mathbf{n} \\ \mathbf{n}\cdot\mathbf{l} & \mathbf{n}\cdot\mathbf{m} & \mathbf{n}\cdot\mathbf{n} \end{vmatrix} = \begin{vmatrix} l^\alpha l_\alpha & l^\alpha m_\alpha & l^\alpha n_\alpha \\ m^\beta l_\beta & m^\beta m_\beta & m^\beta n_\beta \\ n^\gamma l_\gamma & n^\gamma m_\gamma & n^\gamma n_\gamma \end{vmatrix}$$
$$= l^\alpha m^\beta n^\gamma \overline{lmn}_{\alpha\beta\gamma}$$

We obtain five other expressions by permuting $\alpha \quad \beta \quad \gamma$ in all possible ways, add the results, and so have

$$\nu_\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = {}^1/_6\, \overline{lmn}^{\alpha\beta\gamma}\overline{lmn}_{\alpha\beta\gamma} = {}^1/_6\, v^{\alpha\beta\gamma}v_{\alpha\beta\gamma}$$

9. We denote the tetraped **lmnp** by f, the determinant

$$\begin{vmatrix} l^\alpha & l^\beta & l^\gamma & l^\delta \\ m^\alpha & m^\beta & m^\gamma & m^\delta \\ n^\alpha & n^\beta & n^\gamma & n^\delta \\ p^\alpha & p^\beta & p^\gamma & p^\delta \end{vmatrix}$$

by $\overline{lmnp}^{\alpha\beta\gamma\delta}$ and by $f^{\alpha\beta\gamma\delta}$, with the corresponding notation in lower affixes, and have from the expression in article 11,7

$$f = \mathbf{lmnp} = l^\alpha \mathbf{e}_\alpha m^\beta \mathbf{e}_\beta n^\gamma \mathbf{e}_\gamma p^\delta \mathbf{e}_\delta$$

Now $\mathbf{e}_{\alpha\beta\gamma\delta} \equiv \mathbf{e}_\alpha \mathbf{e}_\beta \mathbf{e}_\gamma \mathbf{e}_\delta$ is finite only when $\alpha \quad \beta \quad \gamma \quad \delta$ are all different. So we evaluate the expression for f by giving $\alpha \quad \beta \quad \gamma \quad \delta$ the 24 permutations of 1 2 3 4, and obtain *

$$f = \begin{vmatrix} l^1 & l^2 & l^3 & l^4 \\ m^1 & m^2 & m^3 & m^4 \\ n^1 & n^2 & n^3 & n^4 \\ p^1 & p^2 & p^3 & p^4 \end{vmatrix} \mathbf{e}_{1234} = f^{1234}\sqrt{-g}$$

The corresponding expression in the derived lattice is

$$f = -f_{1234}/\sqrt{-g}$$

To show up the completeness of the symmetry we write $g_{12341234}$ for the $g_{\alpha\beta}$-determinant and $g^{12341234}$ for the $g^{\alpha\beta}$-determinant and have

$$f = f^{1234}\sqrt{-g_{12341234}} = -f_{1234}\sqrt{-g^{12341234}}$$

10. The expression in article 7,5 for the size-number of f gives

$$v_f = \begin{vmatrix} \mathbf{l}\cdot\mathbf{l} & \mathbf{l}\cdot\mathbf{m} & \mathbf{l}\cdot\mathbf{n} & \mathbf{l}\cdot\mathbf{p} \\ \mathbf{m}\cdot\mathbf{l} & \mathbf{m}\cdot\mathbf{m} & \mathbf{m}\cdot\mathbf{n} & \mathbf{m}\cdot\mathbf{p} \\ \mathbf{n}\cdot\mathbf{l} & \mathbf{n}\cdot\mathbf{m} & \mathbf{n}\cdot\mathbf{n} & \mathbf{n}\cdot\mathbf{p} \\ \mathbf{p}\cdot\mathbf{l} & \mathbf{p}\cdot\mathbf{m} & \mathbf{p}\cdot\mathbf{n} & \mathbf{p}\cdot\mathbf{p} \end{vmatrix} = l^\alpha m^\beta n^\gamma p^\delta \begin{vmatrix} l_\alpha & m_\alpha & n_\alpha & p_\alpha \\ l_\beta & m_\beta & n_\beta & p_\beta \\ l_\gamma & m_\gamma & n_\gamma & p_\gamma \\ l_\delta & m_\delta & n_\delta & p_\delta \end{vmatrix}$$

---

* See Appendix on page 165.

We form 24 different expressions from the 24 permutations of $\alpha \; \beta \; \gamma \; \delta$, and add, and have

$$24\nu_f = \overline{lmnp}^{\alpha\beta\gamma\delta}\overline{lmnp}_{\alpha\beta\gamma\delta} = f^{\alpha\beta\gamma\delta}f_{\alpha\beta\gamma\delta}$$

The only finite terms are those for which $\alpha \; \beta \; \gamma \; \delta$ are different, they are given by the 24 permutations of 1 2 3 4, and each is $f^{1234}f_{1234}$, so that

$$\nu_f = f^{1234}f_{1234}$$

an expression which we could have obtained directly by multiplying together the two expressions for f given in the preceding article.

11. **Exercise.**—Of the four vectors that determine a tetraped f let the first two determine the vector area **a** and the other two the vector area **b**. Show that

$$f = (a^{23}b^{14} + a^{31}b^{24} + a^{12}b^{34} + a^{14}b^{23} + a^{24}b^{31} + a^{34}b^{12})\sqrt{-g}$$

12. **Exercise.**—Find the size of $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^3\mathbf{e}^4$ by expressing it in terms of $\mathbf{e}_\alpha$ and also by expressing it in terms of $\mathbf{e}^\alpha$, and so show that $g_{1212} = gg^{3434}$.

More generally by calculating $\mathbf{e}^\alpha\mathbf{e}^\beta\mathbf{e}_\gamma\mathbf{e}_\delta$ show that

$$g_{\kappa\lambda\gamma\delta} = gg^{\alpha\beta\mu\nu}$$

where $\alpha \; \beta \; \gamma \; \delta$ and $\kappa \; \lambda \; \mu \; \nu$ are even permutations from 1 2 3 4.

13. **Exercise.**—Show that $^1/_2\,\Sigma g_{12\alpha\beta}g_{\rho\sigma\gamma\delta}$, summation being for $\alpha \; \beta \; \gamma \; \delta$ which are to have as values the twelve even permutations of 1 2 3 4, is equal to g when $\rho = 3 \;\; \sigma = 4$, equal to $-g$ when $\rho = 4 \;\; \sigma = 3$, and equal to zero in other cases.

r s u v being such vectors that $\overline{rs}_{\alpha\beta} = \sqrt{-g}\;\overline{uv}^{\gamma\delta}$ where $\alpha \; \beta \; \gamma \; \delta$ is an even permutation from 1 2 3 4, show that $\overline{rs}^{\alpha\beta} = -\overline{uv}_{\gamma\delta}/\sqrt{-g}$, $\alpha \; \beta \; \gamma \; \delta$ as before.

CHAPTER 13

THE TENSOR OPERATOR

1. LET $\overrightarrow{OP}$ be a vector joining neighbouring points O P of an elastic solid in Euclidian space. As the result of distortion of the solid let O come to O^1 and P to P^1. Let the distorted solid be given a parallel shift that brings O^1 back to O. An appropriate way to describe the change of configuration of the solid in the neighbourhood of O is by means of an operator which expresses the totality of vectors $\overrightarrow{O^1P^1}$ in terms of the totality $\overrightarrow{OP}$. This operator effects a stretching and is consequently given the Latin name for stretcher, namely "tensor." From this meaning the term tensor is extended to mean an operator in any manifold which effects a linear transformation on the totality of line vectors at a point.

2. A tensor operator is represented in the form

$$\mathbf{kl} + \mathbf{mn} + \dots$$

as the sum of a number of terms each of which contains a pair of line vectors. We use black capital letters to denote tensors; let us call this one **A**. Its method of operation on a vector $\mathbf{x}$ is shown in the equation

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} = \mathbf{k}(\mathbf{l} \cdot \mathbf{x}) + \mathbf{m}(\mathbf{n} \cdot \mathbf{x}) + \dots$$

that is to say the rearward vector of each pair links up with

x in an inner product, the forward vector remaining free. In more concise notation we also write

$$\mathbf{y} = \mathbf{A}_{\mathbf{x}} = \mathbf{k}\varpi_{\mathbf{lx}} + \mathbf{m}\varpi_{\mathbf{nx}} + \ldots$$

This is **A**'s backhand stroke. He has also a forehand stroke whose effect is defined by

$$\mathbf{z} = \mathbf{x} \cdot \mathbf{A} = \varpi_{\mathbf{xk}}\mathbf{l} + \varpi_{\mathbf{xm}}\mathbf{n} + \ldots$$

3. The vector

$$\mathbf{lk} + \mathbf{nm} + \ldots$$

in which each pair of vectors have changed places, is called the conjugate of **A** and is denoted by $\overline{\mathbf{A}}$. $\overline{\mathbf{A}}$'s forehand and backhand strokes have the same effect as **A**'s backhand and forehand strokes, or in symbols

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{A} \qquad \mathbf{x} \cdot \overline{\mathbf{A}} = \mathbf{A} \cdot \mathbf{x}$$

4. Two tensors may be linked together to form another. Thus if $\mathbf{B} = \mathbf{rs} + \mathbf{tu} + \ldots$, the operator which we call the product of **A** and **B** and write as $\mathbf{A} \cdot \mathbf{B}$ is defined by

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{k}\varpi_{\mathbf{lr}}\mathbf{s} + \mathbf{k}\varpi_{\mathbf{lt}}\mathbf{u} + \mathbf{m}\varpi_{\mathbf{nr}}\mathbf{s} + \mathbf{m}\varpi_{\mathbf{nt}}\mathbf{u} + \ldots$$

There are the same number of terms in $\mathbf{A} \cdot \mathbf{B}$ as if the multiplication was in elementary algebra; each pair of vectors that come into contact in the expansion link up into an inner product while the other two vectors remain free to act. At times we write **AB** without the dot instead of $\mathbf{A} \cdot \mathbf{B}$.

5. **Exercise.**—Show that to multiply **A** and **B** together and operate on **x** with the product gives the same result as if **B** produced **y** by operating on **x** and then **A** operated on **y**; that is to say show that

$$\mathbf{AB} \cdot \mathbf{x} = \mathbf{A} \cdot \mathbf{B}_{\mathbf{x}}$$

6. Suppose **A** produces **y** by operating on **x** and then we form the inner product $\mathbf{y} \cdot \mathbf{u}$.

$$\mathbf{y} \cdot \mathbf{u} = \mathbf{A}_{\mathbf{x}} \cdot \mathbf{u} = \varpi_{\mathbf{uk}}\varpi_{\mathbf{lx}} + \varpi_{\mathbf{um}}\varpi_{\mathbf{nx}} + \ . \ . \ .$$

If **A**'s forehand operating on **u** gives **z** and we then form the product $\mathbf{z} \cdot \mathbf{x}$ we have the same result, and

$$\mathbf{A}_{\mathbf{x}} \cdot \mathbf{u} = {}_{\mathbf{u}}\mathbf{A} \cdot \mathbf{x}$$

so that both of them may be written $\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{x}$ or ${}_{\mathbf{u}}\mathbf{A}_{\mathbf{x}}$ in which the order of the two operations is left undetermined.

7. We give each vector in the expression for **A** in terms of its components in the $\mathbf{e}_\alpha$ lattice, and multiply out, taking care to preserve the order between prime vectors. We thus obtain

$$\begin{aligned} \mathbf{A} = {} & a^{11}\mathbf{e}_1\mathbf{e}_1 + a^{12}\mathbf{e}_1\mathbf{e}_2 + a^{13}\mathbf{e}_1\mathbf{e}_3 + a^{14}\mathbf{e}_1\mathbf{e}_4 \\ & + a^{21}\mathbf{e}_2\mathbf{e}_1 + a^{22}\mathbf{e}_2\mathbf{e}_2 + a^{23}\mathbf{e}_2\mathbf{e}_3 + a^{24}\mathbf{e}_2\mathbf{e}_4 \\ & + a^{31}\mathbf{e}_3\mathbf{e}_1 + a^{32}\mathbf{e}_3\mathbf{e}_2 + a^{33}\mathbf{e}_3\mathbf{e}_3 + a^{34}\mathbf{e}_3\mathbf{e}_4 \\ & + a^{41}\mathbf{e}_4\mathbf{e}_1 + a^{42}\mathbf{e}_4\mathbf{e}_2 + a^{43}\mathbf{e}_4\mathbf{e}_3 + a^{44}\mathbf{e}_4\mathbf{e}_4 \end{aligned}$$

in which

$$a^{\alpha\beta} = k^\alpha l^\beta + m^\alpha n^\beta + \ . \ . \ . \ = \Sigma k^\alpha l^\beta$$

That we are at liberty to multiply out in this way is shown by the fact that the new expression has exactly the same effect as the old one when it operates on a vector.

We have thus an expression for **A** which contains 16 independent constants $a^{\alpha\beta}$. The terms $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_1$ of a tensor have quite different effects since $\mathbf{e}_1\mathbf{e}_2$ throws any vector it operates on along $\mathbf{e}_1$ while the other throws vectors along $\mathbf{e}_2$.

8. **Exercise.**—Calculate the effect of operator **A** in the form given in article 7 upon **x**, and show that the result is the same as if **A** was used in the form of article 2.

9. We can also express the tensor operator in terms of the $\mathbf{e}^\alpha$-lattice, and we can express the forward vectors in one

lattice and the rearward vectors in the other. We have thus the four expressions

$$
\begin{aligned}
A &= a^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta \\
&= a_\alpha^{0\beta}\mathbf{e}^\alpha\mathbf{e}_\beta \\
&= a^\alpha_{0\beta}\mathbf{e}_\alpha\mathbf{e}^\beta \\
&= a_{\alpha\beta}\mathbf{e}^\alpha\mathbf{e}^\beta
\end{aligned}
$$

where the zero is introduced into $a_\alpha^{0\beta}$ and $a^\alpha_{0\beta}$ as a buffer to hold the β away and so distinguish these quantities from $a^\beta_{0\alpha}$ and $a_\beta^{0\alpha}$.

10. The values of the coefficients are given by

$$a^{\alpha\beta} = \Sigma k^\alpha l^\beta \qquad a_\alpha^{0\beta} = \Sigma k_\alpha l^\beta \qquad a^\alpha_{0\beta} = \Sigma k^\alpha l_\beta \qquad a_{\alpha\beta} = \Sigma k_\alpha l_\beta$$

From these by use of the expressions that give the derived components of **k l** . . . in terms of the original components and vice versa we obtain

$$a_\alpha^{0\beta} = g_{\alpha\omega}a^{\omega\beta} \qquad a^\alpha_{0\beta} = g_{\beta\omega}a^{\alpha\omega} \qquad a_{\alpha\beta} = g_{\alpha\omega}a^\omega_{0\beta} = g_{\beta\omega}a_\alpha^{0\omega}$$

and

$$a^\alpha_{0\beta} = g^{\alpha\omega}a_{\omega\beta} \qquad a_\alpha^{0\beta} = g^{\beta\omega}a_{\alpha\omega} \qquad a^{\alpha\beta} = g^{\alpha\omega}a_\omega^{0\beta} = g^{\beta\omega}a^\alpha_{0\omega}$$

Thus of the four sets of the quantities any three sets can be expressed in terms of the fourth.

11. The effect of operator

$$\mathbf{I} \equiv \mathbf{e}^\alpha\mathbf{e}_\alpha \equiv \mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 + \mathbf{e}^3\mathbf{e}_3 + \mathbf{e}^4\mathbf{e}_4$$

is shown by

$$
\begin{aligned}
\mathbf{I}\cdot\mathbf{x} &= \mathbf{e}^\alpha x_\alpha = \mathbf{x} \\
\mathbf{x}\cdot\mathbf{I} &= x^\alpha\mathbf{e}_\alpha = \mathbf{x}
\end{aligned}
$$

that is both its forehand and backhand strokes leave the vector **x** unaltered. This operator is called the identity operator.

12. Consider the tensor operator $\mathbf{A} \equiv \mathbf{kl} - \mathbf{lk}$, and suppose that it produces $\mathbf{y}$ by operating on a vector $\mathbf{x}$. Since

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} = \mathbf{k}\varpi_{\mathbf{lx}} - \mathbf{l}\varpi_{\mathbf{kx}}$$

$\mathbf{y}$ lies in the biplane $\mathbf{a} \equiv \mathbf{kl}$. Also since

$$\mathbf{y} \cdot \mathbf{x} = \varpi_{\mathbf{kx}}\varpi_{\mathbf{lx}} - \varpi_{\mathbf{lx}}\varpi_{\mathbf{kx}} = 0$$

$\mathbf{x}$ and $\mathbf{y}$ are at right angles.

Thus the effect of the operation of $\mathbf{A}$ is, apart from any effect it may have on size, to turn the vector through a right angle into the biplane $\mathbf{a}$.

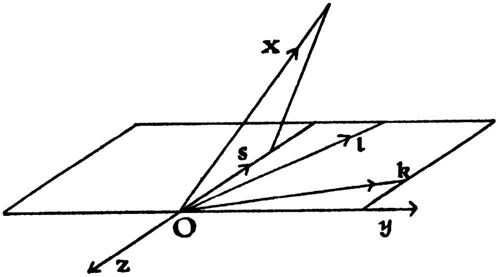


Figure for article 13,12.

If $\mathbf{A}$ now operates on $\mathbf{y}$, it rotates $\mathbf{y}$ through a right angle in the biplane $\mathbf{a}$ to the position $\mathbf{z}$, which is in line with the projection $\mathbf{s}$ of $\mathbf{x}$ on the biplane. We note that our whole figure lies in a triplane, namely that containing $\mathbf{k}$ $\mathbf{l}$ $\mathbf{x}$. Before using this property to calculate the projection $\mathbf{s}$ we must consider the properties of $\mathbf{A}$.

13. We express $\mathbf{k}$ $\mathbf{l}$ in terms of their components, and so have for $\mathbf{A}$ the expressions $a^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta$, etc., where $a^{\alpha\beta}$ $a_{\alpha\beta}$ are $k\bar{l}^{\alpha\beta}$ $\bar{k}\bar{l}_{\alpha\beta}$ the components of the vector parallelogram $\mathbf{a}$ in the prime biplanes of the $\mathbf{e}_\omega$ and $\mathbf{e}^\omega$ lattices. The quantities

$a^{\alpha}_{\circ\beta}$ and $a^{\circ\beta}_{\alpha}$ which are $k^{\alpha}l_{\beta} - l^{\alpha}k_{\beta}$ and $k_{\alpha}l^{\beta} - l_{\alpha}k^{\beta}$ we may call mixed components.

The operator $\mathbf{P} \equiv \mathbf{A} \cdot \mathbf{A}$ has the value

$$\begin{aligned} \mathbf{P} = p^{\rho\sigma}\mathbf{e}_{\rho}\mathbf{e}_{\sigma} &= a^{\alpha\beta}\mathbf{e}_{\alpha}\mathbf{e}_{\beta} \cdot a^{\circ\phi}_{\theta}\mathbf{e}^{\theta}\mathbf{e}_{\phi} \\ &= a^{\alpha\beta}a^{\circ\phi}_{\theta}\delta^{\theta}_{\beta}\mathbf{e}_{\alpha}\mathbf{e}_{\phi} \\ &= a^{\alpha\beta}a^{\circ\phi}_{\beta}\mathbf{e}_{\alpha}\mathbf{e}_{\phi} \end{aligned}$$

so that

$$p^{\rho\sigma} = a^{\rho\omega}a^{\circ\sigma}_{\omega}$$

14. The conjugate tensor $\overline{\mathbf{A}}$ is found from $\mathbf{A}$ by interchanging $\mathbf{e}_{\alpha}$ and $\mathbf{e}_{\beta}$, that is

$$\overline{\mathbf{A}} = a^{\alpha\beta}\mathbf{e}_{\beta}\mathbf{e}_{\alpha}$$

Now $a^{\alpha\beta} = -a^{\beta\alpha}$ since $a^{\alpha\beta}$ are the components of the vector area $\mathbf{a}$, so that

$$\overline{\mathbf{A}} = -a^{\beta\alpha}\mathbf{e}_{\beta}\mathbf{e}_{\alpha} = -\mathbf{A}$$

and for all vectors

$$\mathbf{x} \cdot \mathbf{A} = -\mathbf{A} \cdot \mathbf{x}$$

The tensor $\mathbf{P}$ being $a^{\alpha\omega}a_{\omega\beta}\mathbf{e}_{\alpha}\mathbf{e}^{\beta}$ its conjugate is

$$\overline{\mathbf{P}} = a^{\alpha\omega}a_{\omega\beta}\mathbf{e}^{\beta}\mathbf{e}_{\alpha}$$

or by raising and lowering all the affixes

$$\begin{aligned} \overline{\mathbf{P}} &= a_{\alpha\omega}a^{\omega\beta}\mathbf{e}_{\beta}\mathbf{e}^{\alpha} \\ &= a^{\beta\omega}a_{\omega\alpha}\mathbf{e}_{\beta}\mathbf{e}^{\alpha} = \mathbf{P} \end{aligned}$$

so that $\mathbf{P}$ is self-conjugate, and for every vector $\mathbf{x}$ satisfies the equation

$$\mathbf{x} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{x}$$

15. Also

$$\begin{aligned} \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} = \mathbf{P} \cdot \mathbf{A} &= a_{\alpha\omega}a^{\omega\sigma}\mathbf{e}^{\alpha}\mathbf{e}_{\sigma} \cdot a_{\theta\beta}\mathbf{e}^{\theta}\mathbf{e}^{\beta} \\ &= a_{\alpha\omega}a^{\omega\sigma}a_{\sigma\beta}\mathbf{e}^{\alpha}\mathbf{e}^{\beta} \end{aligned}$$

By interchange of ω σ and addition this becomes

$$1/2\, a^{\omega\sigma}\mathbf{e}^{\alpha}\mathbf{e}^{\beta}(a_{\alpha\omega}a_{\sigma\beta} - a_{\alpha\sigma}a_{\omega\beta})$$

or by article 12,4

$$1/2\, a^{\omega\sigma}a_{\sigma\omega}a_{\alpha\beta}\mathbf{e}^{\alpha}\mathbf{e}^{\beta}$$

which is

$$-\nu_{\mathbf{a}}\mathbf{A}$$

whence also

$$\mathbf{P}\cdot\mathbf{P} = \mathbf{P}\cdot\mathbf{A}\cdot\mathbf{A} = -\nu_{\mathbf{a}}\mathbf{A}\cdot\mathbf{A} = -\nu_{\mathbf{a}}\mathbf{P}$$

16. A tensor operator $\mathbf{A} \equiv a^{\rho\sigma}\mathbf{e}_{\rho}\mathbf{e}_{\sigma}$ for which $a^{\sigma\rho} = a^{\rho\sigma}$ is also called a symmetrical operator, and one for which $a^{\sigma\rho} = -a^{\rho\sigma}$ is called skewsymmetrical.

17. **Exercise.**—Show that if $\mathbf{A} \equiv a^{\rho\sigma}\mathbf{e}_{\rho}\mathbf{e}_{\sigma}$ is a skewsymmetrical operator and $a^{23}a^{14} + a^{31}a^{24} + a^{12}a^{34} = 0$, the operator can be expressed in terms of two line vectors $\mathbf{l}$ $\mathbf{m}$ in the form $\mathbf{A} = \mathbf{lm} - \mathbf{ml}$.

Projection of one Line Vector on another.

18. Let $\mathbf{r}$ $\mathbf{s}$ be the projection and the projector of the vector $\mathbf{x}$ on the vector $\mathbf{k}$. We put $\mathbf{r} = \kappa\mathbf{k}$, and the relation $\mathbf{r}\cdot\mathbf{r} = \mathbf{r}\cdot\mathbf{x}$ gives $\kappa^2\mathbf{k}\cdot\mathbf{k} = \kappa\mathbf{k}\cdot\mathbf{x}$ and

$$\mathbf{r} = \mathbf{k}\varpi_{\mathbf{kx}}/\nu_{\mathbf{k}}$$

We take the operator $\mathbf{Q} \equiv \mathbf{kk}/\nu_{\mathbf{k}}$ and have

$$\mathbf{r} = \mathbf{Q}\cdot\mathbf{x}$$

so that operator $\mathbf{Q}$ converts $\mathbf{x}$ into its projection.

And since $\mathbf{s} = \mathbf{x} - \mathbf{r}$ the operator $\mathbf{I} - \mathbf{Q}$ converts $\mathbf{x}$ into its projector, $\mathbf{I}$ being the identity operator.

CHAPTER 14

PROJECTION OF ONE AREA VECTOR ON ANOTHER

1. WE consider the projection of a line vector $\mathbf{x}$ on a vector parallelogram $\mathbf{a} \equiv \mathbf{kl}$. We denote the projection of $\mathbf{x}$ by $\mathbf{s}$, and denote by $\mathbf{p}$ the projector, that is the vector which runs from the head of $\mathbf{s}$ to the head of $\mathbf{x}$.

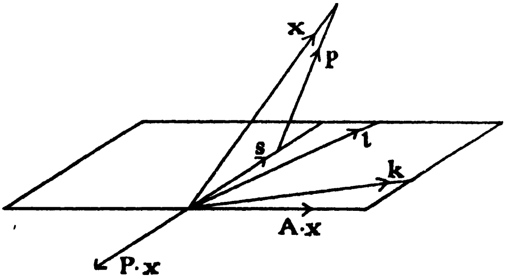


Figure for article 14,1.

We use

$\mathbf{A} \equiv a^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta$ to denote the operator $\mathbf{kl} - \mathbf{lk} \equiv \overline{kl}^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta$
$\mathbf{C} \equiv c^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta$ „ „ $\mathbf{A}\cdot\mathbf{A}$
$\mathbf{D} \equiv d^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta$ „ „ $\mathbf{C} + \nu_{\mathbf{a}}\mathbf{I}$

2. When $\mathbf{A}$ operates on $\mathbf{x}$ the resultant is

$$\mathbf{A}\cdot\mathbf{x} = (\mathbf{kl} - \mathbf{lk})\cdot\mathbf{x} = \mathbf{k}\varpi_{\mathbf{lx}} - \mathbf{l}\varpi_{\mathbf{kx}}$$

and the inner product of the resultant and $\mathbf{x}$ is

$$\mathbf{x}\cdot\mathbf{A}\cdot\mathbf{x} = \varpi_{\mathbf{kx}}\varpi_{\mathbf{lx}} - \varpi_{\mathbf{lx}}\varpi_{\mathbf{kx}} = 0$$

Thus the effect of operator $\mathbf{A}$ is to throw $\mathbf{x}$ into the biplane $\mathbf{a}$ in a position at right angles to $\mathbf{x}$. Hence if we operate a second time with $\mathbf{A}$ we get a vector lying in $\mathbf{a}$ and at right angles to $\mathbf{A}\cdot\mathbf{x}$, that is to say a vector which lies in the same line as $\mathbf{s}$, the projection of $\mathbf{x}$.

3. We use $\mathbf{z}$ to denote the vector $\mathbf{C}\cdot\mathbf{x} \equiv \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{x}$ which lies along $\mathbf{s}$. We have now with the help of article 13,15

$$\mathbf{z}\cdot\mathbf{z} = \mathbf{x}\cdot\mathbf{C}\cdot\mathbf{C}\cdot\mathbf{x} = \mathbf{x}\cdot(-\nu_{\mathbf{a}}\mathbf{C})\cdot\mathbf{x} = -\nu_{\mathbf{a}}\mathbf{x}\cdot\mathbf{z} = -\nu_{\mathbf{a}}\mathbf{s}\cdot\mathbf{z}$$

so that

$$\mathbf{s} = -\mathbf{z}/\nu_{\mathbf{a}} = (-\mathbf{C}/\nu_{\mathbf{a}})\cdot\mathbf{x}$$

and the operator $-\mathbf{C}/\nu_{\mathbf{a}}$ converts $\mathbf{x}$ into its projection.

4. The projector $\mathbf{p}$ is equal to

$$\mathbf{x} - \mathbf{s} = (\mathbf{C}/\nu_{\mathbf{a}} + \mathbf{I})\cdot\mathbf{x} = (\mathbf{D}/\nu_{\mathbf{a}})\cdot\mathbf{x}$$

so that the operator $\mathbf{D}/\nu_{\mathbf{a}}$ converts $\mathbf{x}$ into its projector.

5. Now let the vector parallelogram $\mathbf{b} \equiv \mathbf{xy}$ project into $\mathbf{st}$, $\mathbf{x}$ projecting to $\mathbf{s}$ and $\mathbf{y}$ to $\mathbf{t}$. We have

$$-\nu_{\mathbf{a}}\mathbf{s} = \mathbf{C}\cdot\mathbf{x} = c^{\alpha\beta}\mathbf{e}_{\alpha}x_{\beta} \qquad -\nu_{\mathbf{a}}\mathbf{t} = c^{\theta\phi}\mathbf{e}_{\theta}y_{\phi}$$

so that

$$\nu_{\mathbf{a}}^2\mathbf{st} = c^{\alpha\beta}c^{\theta\phi}x_{\beta}y_{\phi}\mathbf{e}_{\alpha\theta}$$

Interchange of $\alpha\ \beta$ with $\theta\ \phi$ and addition gives for the right side

$$\tfrac{1}{2}\, c^{\alpha\beta}c^{\theta\phi}b_{\beta\phi}\mathbf{e}_{\alpha\theta}$$

From this the interchange of β with ϕ and addition gives

$$\tfrac{1}{4}\begin{vmatrix} c^{\alpha\beta} & c^{\alpha\phi} \\ c^{\theta\beta} & c^{\theta\phi} \end{vmatrix} b_{\beta\phi}\mathbf{e}_{\alpha\theta}$$

6. Now this determinant has the value

$$\begin{vmatrix} a^{\alpha\rho}a^{\circ\beta}_{\rho} & a^{\alpha\sigma}a^{\circ\phi}_{\sigma} \\ a^{\theta\rho}a^{\circ\beta}_{\rho} & a^{\theta\sigma}a^{\circ\phi}_{\sigma} \end{vmatrix}$$

$$= a^{\circ\beta}_{\rho}a^{\circ\phi}_{\sigma}\begin{vmatrix} a^{\alpha\rho} & a^{\alpha\sigma} \\ a^{\theta\rho} & a^{\theta\sigma} \end{vmatrix}$$

or by article 12,4

$$= a^{\circ\beta}_{\rho}a^{\circ\phi}_{\sigma}a^{\alpha\theta}a^{\rho\sigma}$$

Hence

$$\nu_{\mathbf{a}}^2\mathbf{st} = {}^1/_4\, a_{\rho}^{\circ\beta} a_{\sigma}^{\circ\phi} a^{\rho\sigma} b_{\beta\phi} a^{\alpha\theta} \mathbf{e}_{\alpha\theta}$$

or by raising and lowering certain affixes

$$= {}^1/_2\, a^{\rho\beta} a^{\sigma\phi} a_{\rho\sigma} b_{\beta\phi} \mathbf{a}$$

7. The interchange of β with ϕ and addition gives for the coefficient of **a**

$$^1/_4 \begin{vmatrix} a^{\rho\beta} & a^{\rho\phi} \\ a^{\sigma\beta} & a^{\sigma\phi} \end{vmatrix} a_{\rho\sigma} b_{\beta\phi}$$

$$= {}^1/_4\, a^{\rho\sigma} a^{\beta\phi} a_{\rho\sigma} b_{\beta\phi}$$

$$= \nu_{\mathbf{a}} \varpi_{\mathbf{ab}}$$

where $\varpi_{\mathbf{ab}}$ means ${}^1/_2\, a^{\alpha\beta} b_{\alpha\beta}$.

Hence $\nu_{\mathbf{a}}\mathbf{st} = \varpi_{\mathbf{ab}}\mathbf{a}$.

We observe that **st** and **a** face the same way if $\nu_{\mathbf{a}}$ and $\varpi_{\mathbf{ab}}$ are both positive or both negative, and face opposite ways if one is positive and one negative.

8. We have also

$$\nu_{\mathbf{a}} \sigma_{\mathbf{st}} = \pm\, \varpi_{\mathbf{ab}} \sigma_{\mathbf{a}}$$

the + or − to be used according as $\nu_{\mathbf{a}}$ $\varpi_{\mathbf{ab}}$ have the same or opposite signs, and

$$\sigma_{\mathbf{a}} \sigma_{\mathbf{st}} = \pm\, \varpi_{\mathbf{ab}}$$

the + or − to be used according as $\varpi_{\mathbf{ab}}$ is positive or negative; and (clear of ambiguous signs)

$$\nu_{\mathbf{a}} \nu_{\mathbf{st}} = \varpi_{\mathbf{ab}}^2$$

The symmetry of the results shows that the product of the size of **b** and the size of the projection of **a** on **b** is equal to the product of the size of **a** and the size of the projection of **b** on **a**.

The quantity $\varpi_{\mathbf{ab}}$ is called the inner product of the two vector areas **a** **b**. It is also written in the form $\mathbf{a} \cdot \mathbf{b}$.

9. **Exercise.**—How are two area vectors $\mathbf{a} \quad \mathbf{b}$ related to one another when $\varpi_{\mathbf{ab}} = 0$?

10. **Exercise.**—Show that if $\mathbf{a}$ is an area vector

$$\varpi_{\mathbf{a}\mathbf{e}_{\alpha\beta}} = a_{\alpha\beta} \qquad \varpi_{\mathbf{a}\mathbf{e}^{\alpha\beta}} = a^{\alpha\beta}$$

and that the projection of $\mathbf{a}$ on $\mathbf{e}_{12}$ is $a_{12}\mathbf{e}_{12}/g_{1212}$ so that $a_{\alpha\beta}$ besides being the components in the $\mathbf{e}^{\omega}$ lattice, are (except for constant scalar multipliers like g_{1212}) the orthogonal projections in the $\mathbf{e}_{\omega}$ lattice.

11. **Exercise.**—Join the heads of the vectors $\mathbf{x} \quad \mathbf{y} \quad \mathbf{s} \quad \mathbf{t}$ of article 5 by straight lines, and fill in the quadrilateral so formed by a film made up of two plane triangles. The four triangles that stand on the four sides of the quadrilateral and have the origin for vertex, together with the film, form a closed bisurface. Express the six area vectors that form this bisurface looking inwards, in terms of $\mathbf{x} \quad \mathbf{y} \quad \mathbf{s} \quad \mathbf{t}$, and verify that their sum is zero.

12. **Exercise.**—Show that $\mathbf{e}_{\alpha\beta} \cdot \mathbf{e}_{\gamma\delta} = g_{\alpha\beta\gamma\delta}$ and that $\mathbf{e}_{12} \cdot \mathbf{e}^{\rho\sigma}$ is equal to 1 if $\rho = 1 \quad \sigma = 2$, equal to -1 if $\rho = 2 \quad \sigma = 1$, and is zero for all other values of $\rho \quad \sigma$.

CHAPTER 15

PROJECTION OF ONE VOLUME VECTOR ON ANOTHER

1. LET a line vector $\mathbf{x}$ project on the triplane normal to line vector $\mathbf{n}$. Let its projection be $\mathbf{s}$ and its projector be $\mathbf{p}$, so that $\mathbf{x} = \mathbf{s} + \mathbf{p}$.

We denote the tensor operator $\mathbf{nn}$ by $\mathbf{N}$, and the operator $\nu_{\mathbf{n}}\mathbf{I} - \mathbf{N}$ by $\mathbf{C}$. We note that $\mathbf{N} \bullet \mathbf{N}$ is equal to $\mathbf{n}(\mathbf{n} \bullet \mathbf{n})\mathbf{n}$ or $\nu_{\mathbf{n}}\mathbf{N}$.

If we put $\mathbf{z} = \mathbf{N} \bullet \mathbf{x} = \mathbf{n}\varpi_{\mathbf{nx}}$ we have

$$\mathbf{z} \bullet \mathbf{z} = \mathbf{x} \bullet \mathbf{N} \bullet \mathbf{N} \bullet \mathbf{x} = \mathbf{x} \bullet \nu_{\mathbf{n}}\mathbf{N} \bullet \mathbf{x} = \nu_{\mathbf{n}}\mathbf{x} \bullet \mathbf{z} = \nu_{\mathbf{n}}\mathbf{p} \bullet \mathbf{z}$$

so that

$$\nu_{\mathbf{n}}\mathbf{p} = \mathbf{z} = \mathbf{N} \bullet \mathbf{x}$$

and the operator $\mathbf{N}/\nu_{\mathbf{n}}$ converts $\mathbf{x}$ into its projector $\mathbf{p}$.

2. We have also

$$\mathbf{s} = \mathbf{x} - \mathbf{p} = (\mathbf{I} - \mathbf{N}/\nu_{\mathbf{n}}) \bullet \mathbf{x} = (\mathbf{C}/\nu_{\mathbf{n}}) \bullet \mathbf{x}$$

so that the operator $\mathbf{C}/\nu_{\mathbf{n}}$ converts $\mathbf{x}$ into its projection $\mathbf{s}$.

3. Let area vector $\mathbf{a} = \mathbf{xy}$ be projected on the triplane normal to $\mathbf{n}$, $\mathbf{x}$ having projector $\mathbf{p}$ and projection $\mathbf{s}$, $\mathbf{y}$ having projector $\mathbf{q}$ and projection $\mathbf{t}$. Let *X Y S T* denote the heads of the vectors $\mathbf{x}$ $\mathbf{y}$ $\mathbf{s}$ $\mathbf{t}$. $\mathbf{s}$ and $\mathbf{t}$ are given by

$$\nu_{\mathbf{n}}\mathbf{s} = \mathbf{C} \bullet \mathbf{x} = \nu_{\mathbf{n}}\mathbf{x} - \mathbf{n}\varpi_{\mathbf{nx}}$$
$$\nu_{\mathbf{n}}\mathbf{t} = \mathbf{C} \bullet \mathbf{y} = \nu_{\mathbf{n}}\mathbf{y} - \mathbf{n}\varpi_{\mathbf{ny}}$$

Hence vector area **st** is given by

$$\begin{aligned}\nu_n^2\mathbf{st} &= (\nu_n\mathbf{x} - \mathbf{n}\varpi_{nx})(\nu_n\mathbf{y} - \mathbf{n}\varpi_{ny})\\ &= \nu_n^2\mathbf{xy} - \nu_n\mathbf{xn}\varpi_{ny} - \mathbf{ny}\nu_n\varpi_{nx}\\ \nu_n\mathbf{st} &= \nu_n\mathbf{xy} - \mathbf{xn}\varpi_{ny} + \mathbf{yn}\varpi_{nx}\end{aligned}$$

We denote operator **xy** − **yx** by **A**. Then

$$\mathbf{A}\cdot\mathbf{n} = \mathbf{x}\varpi_{yn} - \mathbf{y}\varpi_{xn}$$

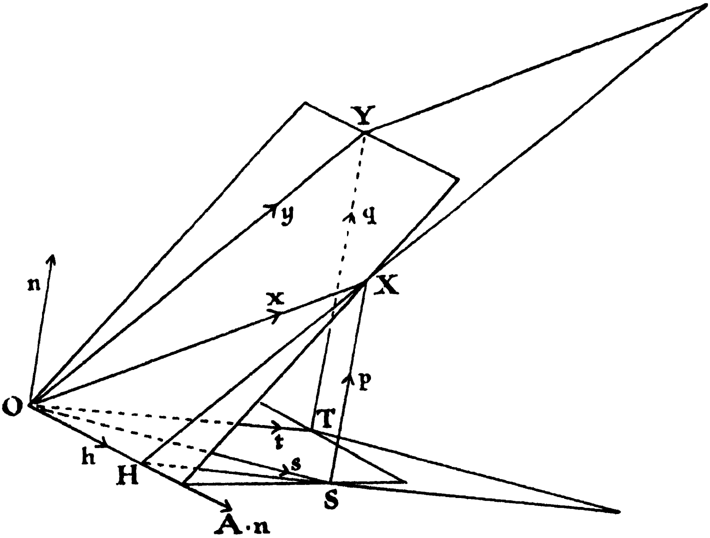


Figure for article 15,3.

and we have

$$\nu_n\mathbf{st} = \nu_n\mathbf{xy} - \mathbf{A}_n\mathbf{n}$$

or

$$\mathbf{xy} = \mathbf{st} + \mathbf{A}_n\mathbf{n}/\nu_n$$

4. Our whole figure lies in a triplane, namely that determined by the vectors **x** **y** **n**. Hence the two biplanes **xy** **st** meet in a line.

Through vector **p** we draw a biplane parallel to vector **y**.

Let it cut the line of intersection of **xy** **st** in *H*, and let the vector $\overrightarrow{OH}$ be called **h**.

Consider now the triangular prism of which the vectors **y** **t** **q** form one end and **h** the height. The parallelogram faces taken as looking inward have a zero vector sum, and are

$$-\mathbf{hy} \quad \mathbf{ht} \quad \mathbf{hq}$$

Since the parallelogram **hy** has the same size-number, orientation, and aspect as **xy** we may write the operator **A** in the form

$$\mathbf{A} = \mathbf{hy} - \mathbf{yh}$$

which gives

$$\mathbf{A} \bullet \mathbf{n} = \mathbf{h}\varpi_{\mathbf{yn}} - \mathbf{y}\varpi_{\mathbf{hn}} = \mathbf{h}\varpi_{\mathbf{yn}}$$

Also

$$\nu_{\mathbf{n}}\mathbf{q} = \mathbf{N} \bullet \mathbf{y} = \mathbf{n}\varpi_{\mathbf{ny}}$$

so that

$$\mathbf{hq} = \mathbf{A}_{\mathbf{n}}\mathbf{n}/\nu_{\mathbf{n}}$$

Parallelogram **hy** is vectorially equal to **xy**, and parellelogram **ht** to **st**. Thus the vanishing of the vector sum $-\mathbf{hy} + \mathbf{ht} + \mathbf{hq}$ is a way of stating that

$$-\mathbf{xy} + \mathbf{st} + \mathbf{A}_{\mathbf{n}}\mathbf{n}/\nu_{\mathbf{n}} = 0$$

and the equation of article 3 has been proved geometrically.

5. Let a triped **xyz** project into a triplane perpendicular to the vector **k**, **s** **t** **u** being the projections of **x** **y** **z**. The equation of article 2 gives

$$\nu_{\mathbf{k}}{}^3\mathbf{stu} = (\nu_{\mathbf{k}}\mathbf{x} - \varpi_{\mathbf{kx}}\mathbf{k})(\nu_{\mathbf{k}}\mathbf{y} - \varpi_{\mathbf{ky}}\mathbf{k})(\nu_{\mathbf{k}}\mathbf{z} - \varpi_{\mathbf{kz}}\mathbf{k})$$

Expanding we have, since $\mathbf{kk} = 0$,

$$\nu_{\mathbf{k}}\mathbf{stu} = \nu_{\mathbf{k}}\mathbf{xyz} - \varpi_{\mathbf{kx}}\mathbf{kyz} - \varpi_{\mathbf{ky}}\mathbf{xkz} - \varpi_{\mathbf{kz}}\mathbf{xyk}$$

For the first term on the right we put $k_\omega \mathbf{e}_{\alpha\beta\gamma} k^\omega x^\alpha y^\beta z^\gamma$. Then adding the six quantities found by permuting $\alpha\ \beta\ \gamma$ we find as its value $1/6\, k^\omega \overline{xyz}^{\alpha\beta\gamma} k_\omega \mathbf{e}_{\alpha\beta\gamma}$. Similarly for the other terms we have

$$
\begin{aligned}
-\varpi_{\mathbf{kx}}\mathbf{kyz} &= -k_\omega \mathbf{e}_{\alpha\beta\gamma} x^\omega k^\alpha y^\beta z^\gamma &&= -1/6\, x^\omega \overline{kyz}^{\alpha\beta\gamma} k_\omega \mathbf{e}_{\alpha\beta\gamma} \\
-\varpi_{\mathbf{ky}}\mathbf{xkz} &= -k_\omega \mathbf{e}_{\alpha\beta\gamma} y^\omega x^\alpha k^\beta z^\gamma &&= -1/6\, y^\omega \overline{xkz}^{\alpha\beta\gamma} k_\omega \mathbf{e}_{\alpha\beta\gamma} \\
-\varpi_{\mathbf{kz}}\mathbf{xyk} &= -k_\omega \mathbf{e}_{\alpha\beta\gamma} z^\omega x^\alpha y^\beta k^\gamma &&= -1/6\, z^\omega \overline{xyz}^{\alpha\beta\gamma} k_\omega \mathbf{e}_{\alpha\beta\gamma}
\end{aligned}
$$

Now $\overline{kxyz}^{\omega\alpha\beta\gamma} = k^\omega \overline{xyz}^{\alpha\beta\gamma} - x^\omega \overline{kyz}^{\alpha\beta\gamma} + y^\omega \overline{kxz}^{\alpha\beta\gamma} - z^\omega \overline{kxy}^{\alpha\beta\gamma}$ so that

$$\nu_{\mathbf{k}}\mathbf{stu} = 1/6\, \overline{kxyz}^{\omega\alpha\beta\gamma} k_\omega \mathbf{e}_{\alpha\beta\gamma}$$

6. Consider the terms of this expression in $\mathbf{e}_{234}$. As the determinant is finite only when $\alpha\ \beta\ \gamma\ \omega$ are all different, ω must be 1. We give $\beta\ \gamma\ \omega$ the six permutations of 2 3 4, and so have

$$\overline{kxyz}^{1234} k_1 \mathbf{e}_{234}$$

Hence

$$
\begin{aligned}
(k^1 k_1 + k^2 k_2 + k^3 k_3 + k^4 k_4)\mathbf{stu} \\
= \overline{kxyz}^{1234}(k_1 \mathbf{e}_{234} + k_2 \mathbf{e}_{143} + k_3 \mathbf{e}_{412} + k_4 \mathbf{e}_{321}).
\end{aligned}
$$

7. Let a volume vector normal to $\mathbf{k}$ be

$$\mathbf{v} \equiv 1/6\, v^{\alpha\beta\gamma} \mathbf{e}_{\alpha\beta\gamma} \equiv 1/6\, v_{\alpha\beta\gamma} \mathbf{e}^{\alpha\beta\gamma}$$

Then by articles 11,11 and 11,12

$$
\begin{aligned}
k^1/v_{234} &= k^2/v_{143} = k^3/v_{412} = k^4/v_{321} \\
k_1/v^{234} &= k_2/v^{143} = k_3/v^{412} = k_4/v^{321}
\end{aligned}
$$

The equation of article 6 being linear in k^ω and also in k_ω we may replace them by the corresponding $\mathbf{v}$-expressions. We also write $\mathbf{xyz} = 1/6\, \overline{xyz}^{\alpha\beta\gamma} \mathbf{e}_{\alpha\beta\gamma} = \mathbf{w} = 1/6\, w^{\alpha\beta\gamma} \mathbf{e}_{\alpha\beta\gamma}$. The determinant $\overline{kxyz}^{1234}$ is thus replaced by

$$v_{234} w^{234} + v_{143} w^{143} + v_{412} w^{412} + v_{321} w^{321}$$

which we denote by

$$\mathbf{v} \bullet \mathbf{w} \text{ or } \varpi_{\mathbf{vw}}$$

and the equation becomes

$$\nu_{\mathbf{v}}\mathbf{stu} = \varpi_{\mathbf{vw}}\mathbf{v}$$

8. From the result of the preceding article we have

$$\nu_{\mathbf{v}}\sigma_{\mathbf{stu}} = \pm\, \varpi_{\mathbf{vw}}\sigma_{\mathbf{v}}$$

the upper sign to be taken if $\nu_{\mathbf{v}}$ $\varpi_{\mathbf{vw}}$ have the same sign, the lower sign to be taken if they have opposite signs, or

$$\sigma_{\mathbf{v}}\sigma_{\mathbf{stu}} = \pm\, \varpi_{\mathbf{vw}}$$

the upper or lower sign to be used according as $\varpi_{\mathbf{vw}}$ is positive or negative. We can also write the result, in a form free from ambiguity of sign, as

$$\nu_{\mathbf{v}}\nu_{\mathbf{stu}} = \varpi^2_{\mathbf{vw}}$$

The symmetry of these results shows that the product of the size of **w** and the size of the projection of **v** on **w** is equal to the product of the size of **v** and the size of the projection of **w** on **v**.

The quantity $\varpi_{\mathbf{vw}}$ is called the inner product of the two vector volumes **v** **w**.

9. The projection of one volume vector on another can be otherwise treated. The projection of line vector **x** on triped **v** which is normal to **n** is given (article 15,2) by

$$\mathbf{s} = (\mathbf{C}/\nu_{\mathbf{n}}) \bullet \mathbf{x}$$

where **C** is the operator

$$\mathbf{C} = c_{\alpha}^{\circ\beta}\mathbf{e}^{\alpha}\mathbf{e}_{\beta} = \nu_{\mathbf{n}}\mathbf{I} - \mathbf{N} = \nu_{\mathbf{n}}\mathbf{e}^{\beta}\mathbf{e}_{\beta} - n_{\alpha}n^{\beta}\mathbf{e}^{\alpha}\mathbf{e}_{\beta} = (\nu_{\mathbf{n}}\delta_{\alpha}^{\beta} - n_{\alpha}n^{\beta})\mathbf{e}^{\alpha}\mathbf{e}_{\beta}$$

so that $c_{\alpha}^{\circ\beta} = \nu_{\mathbf{n}}\delta_{\alpha}^{\beta} - n_{\alpha}n^{\beta}$.

We introduce four vectors $\mathbf{h}^\omega$ defined by

$$\mathbf{h}^\omega = c_\alpha^{o\omega}\mathbf{e}^\alpha = \nu_{\mathbf{n}}\mathbf{e}^\omega - n^\omega\mathbf{n}$$

Then $$\mathbf{C} = \mathbf{h}^\omega\mathbf{e}_\omega$$

and $$\nu_{\mathbf{n}}\mathbf{s} = \mathbf{C}\bullet\mathbf{x} = x_\omega\mathbf{h}^\omega$$

If $\mathbf{s}\ \ \mathbf{t}\ \ \mathbf{u}$ are the projections of $\mathbf{x}\ \ \mathbf{y}\ \ \mathbf{z}$, this last equation gives

$$\nu_{\mathbf{n}}^3\mathbf{stu} = x_\alpha y_\beta z_\gamma\mathbf{h}^\alpha\mathbf{h}^\beta\mathbf{h}^\gamma$$

or by taking the six permutations of $\alpha\ \ \beta\ \ \gamma$ and adding

$${}^1/6\ w_{\alpha\beta\gamma}\mathbf{h}^\alpha\mathbf{h}^\beta\mathbf{h}^\gamma$$

or in full

$$w_{234}\mathbf{h}^2\mathbf{h}^3\mathbf{h}^4 + w_{143}\mathbf{h}^1\mathbf{h}^4\mathbf{h}^3 + w_{412}\mathbf{h}^4\mathbf{h}^1\mathbf{h}^2 + w_{321}\mathbf{h}^3\mathbf{h}^2\mathbf{h}^1$$

The definition of $\mathbf{h}^\omega$ gives

$$\begin{aligned}\mathbf{h}^2\mathbf{h}^3\mathbf{h}^4 &= (\nu_{\mathbf{n}}\mathbf{e}^2 - n^2\mathbf{n})(\nu_{\mathbf{n}}\mathbf{e}^3 - n^3\mathbf{n})(\nu_{\mathbf{n}}\mathbf{e}^4 - n^4\mathbf{n})\\ &= \nu_{\mathbf{n}}^3\mathbf{e}^2\mathbf{e}^3\mathbf{e}^4 - \nu_{\mathbf{n}}^2 n^2\mathbf{n}\mathbf{e}^3\mathbf{e}^4 - \nu_{\mathbf{n}}^2 n^3\mathbf{e}^2\mathbf{n}\mathbf{e}^4 - \nu_{\mathbf{n}}^2 n^4\mathbf{e}^2\mathbf{e}^3\mathbf{n}\\ &= \nu_{\mathbf{n}}^2\{\nu_{\mathbf{n}}\mathbf{e}^2\mathbf{e}^3\mathbf{e}^4 - n^2(n_1\mathbf{e}^1 + n_2\mathbf{e}^2)\mathbf{e}^3\mathbf{e}^4 - n^3\mathbf{e}^2(n_1\mathbf{e}^1 + n_3\mathbf{e}^3)\mathbf{e}^4\\ &\qquad\qquad - n^4\mathbf{e}^2\mathbf{e}^3(n_1\mathbf{e}^1 + n_4\mathbf{e}^4)\}\\ &= \nu_n^2 n_1(n^1\mathbf{e}^{234} + n^2\mathbf{e}^{143} + n^3\mathbf{e}^{412} + n^4\mathbf{e}^{321})\end{aligned}$$

and so for the other terms. Hence

$$\begin{aligned}n_\omega n^\omega\mathbf{stu} = {}&(n_1w_{234} + n_2w_{143} + n_3w_{412} + n_4w_{321})\\ &\qquad(n^1\mathbf{e}^{234} + n^2\mathbf{e}^{143} + n^3\mathbf{e}^{412} + n^4\mathbf{e}^{321})\end{aligned}$$

which is the equation of article 6 and gives as before

$$\nu_{\mathbf{v}}\mathbf{stu} = \varpi_{\mathbf{vw}}\mathbf{v}$$

10. We observe that $\mathbf{p}\ \ \mathbf{q}\ \ \mathbf{r}$ of article 11,26 can be expressed as $\mathbf{n}\varpi_{\mathbf{xn}}/\nu_{\mathbf{n}}\quad \mathbf{n}\varpi_{\mathbf{yn}}/\nu_{\mathbf{n}}\quad \mathbf{n}\varpi_{\mathbf{zn}}/\nu_{\mathbf{n}}$
so that the equation of that article, and of article 11,29, is equivalent to the equation

$$\nu_{\mathbf{k}}\mathbf{stu} = \nu_{\mathbf{k}}\mathbf{xyz} - \varpi_{\mathbf{kx}}\mathbf{kyz} - \varpi_{\mathbf{ky}}\mathbf{xkz} - \varpi_{\mathbf{kz}}\mathbf{xyk}$$

of article 15,5.

11. **Exercise.**—How are two vector volumes $\mathbf{v}$ $\mathbf{w}$ related to one another if $\mathbf{v} \cdot \mathbf{w} = 0$?

12. **Exercise.**—Show that, $\mathbf{v}$ being any vector volume

$$\mathbf{v} \cdot \mathbf{e}_{\alpha\beta\gamma} = v_{\alpha\beta\gamma} \qquad \mathbf{v} \cdot \mathbf{e}^{\alpha\beta\gamma} = v^{\alpha\beta\gamma}$$

and that the projection of $\mathbf{v}$ on $\mathbf{e}_{123}$ is $v_{123}\mathbf{e}_{123}/g_{123123}$, so that $v_{\alpha\beta\gamma}$ besides being the components of $\mathbf{v}$ in the e^{α} lattice are (except for constant scalar multipliers like g_{123123}) the orthogonal projections of $\mathbf{v}$ in the e_{α} lattice.

CHAPTER 16

TRANSFORMATION TO A NEW REFERENCE LATTICE

1. In addition to the **e**-lattice with prime vectors $\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\ \mathbf{e}_4$ we introduce another, the **c**-lattice with the same origin and with $\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \mathbf{c}_4$ as prime vectors. The prime vectors of each lattice can be expressed in terms of the other set. Let these expressions be

$$\mathbf{c}_\alpha = d^{\circ\rho}_{\alpha}\mathbf{e}_\rho \qquad \mathbf{e}_\alpha = f^{\circ\rho}_{\alpha}\mathbf{c}_\rho$$

there being of course 16 constants $d^{\circ\rho}_{\alpha}$ and 16 constants $f^{\circ\rho}_{\alpha}$.

2. The combination of these expressions gives

$$\mathbf{c}_\alpha = d^{\circ\rho}_{\alpha}\mathbf{e}_\rho = d^{\circ\rho}_{\alpha}f^{\circ\sigma}_{\rho}\mathbf{c}_\sigma$$

The coefficient of $\mathbf{c}_\alpha$ on the right must be unity and the other coefficients zero, that is to say

$$d^{\circ\rho}_{\alpha}f^{\circ\sigma}_{\rho} = \delta^{\sigma}_{\alpha}$$

In this there are 16 equations which give the $d^{\circ\beta}_{\alpha}$ in terms of the $f^{\circ\beta}_{\alpha}$ and vice versa. From

$$\mathbf{e}_\alpha = f^{\circ\rho}_{\alpha}\mathbf{c}_\rho = f^{\circ\rho}_{\alpha}d^{\circ\sigma}_{\rho}\mathbf{e}_\sigma$$

we obtain the 16 equations in another form, namely

$$f^{\circ\rho}_{\alpha}d^{\circ\sigma}_{\rho} = \delta^{\sigma}_{\alpha}$$

3. We define the symbols $d \quad d_{\beta\gamma\delta}{}^{\phi\psi\omega} \quad D_\theta^{\circ\alpha}$ as follows:—

$$d = \begin{vmatrix} d_1^{\circ 1} & d_2^{\circ 1} & d_3^{\circ 1} & d_4^{\circ 1} \\ d_1^{\circ 2} & d_2^{\circ 2} & d_3^{\circ 2} & d_4^{\circ 2} \\ d_1^{\circ 3} & d_2^{\circ 3} & d_3^{\circ 3} & d_4^{\circ 3} \\ d_1^{\circ 4} & d_2^{\circ 4} & d_3^{\circ 4} & d_4^{\circ 4} \end{vmatrix},$$

$$d_{\beta\gamma\delta}{}^{\phi\psi\omega} = \begin{vmatrix} d_\beta^{\circ\phi} & d_\gamma^{\circ\phi} & d_\delta^{\circ\phi} \\ d_\beta^{\circ\psi} & d_\gamma^{\circ\psi} & d_\delta^{\circ\psi} \\ d_\beta^{\circ\omega} & d_\gamma^{\circ\omega} & d_\delta^{\circ\omega} \end{vmatrix}$$

$$dD_\theta^{\circ\alpha} = d_{\beta\gamma\delta}{}^{\phi\psi\omega}$$

n which $\alpha \quad \beta \quad \gamma \quad \delta$ and $\theta \quad \phi \quad \psi \quad \omega$ are even permutations of $1 \quad 2 \quad 3 \quad 4$.

We observe that $d_\rho^{\circ\alpha} dD_\beta^{\circ\rho}$ gives an expansion of the determinant d when α and β are equal, and vanishes when they re unequal, so that

$$D_\beta^{\circ\rho} d_\rho^{\circ\alpha} = \delta_\beta^\alpha$$

These sixteen linear equations give the sixteen quantities $D_\alpha^{\circ\beta}$ in terms of $d_\alpha^{\circ\beta}$, and they are the same equations found in article 2 for $f_\alpha^{\circ\beta}$, so that the two sets of solutions are identical; that is

$$D_\alpha^{\circ\beta} = f_\alpha^{\circ\beta}$$

and
$$d_{\beta\gamma\delta}{}^{\phi\psi\omega} = df_\theta^{\circ\alpha}$$

$\alpha \quad \beta \quad \gamma \quad \delta$ and $\theta \quad \phi \quad \psi \quad \omega$ being even permutations.

4. The new prime tetraped $\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4$ is given in terms o the old by

$$\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4 = d_1^{\circ\alpha}\mathbf{e}_\alpha d_2^{\circ\beta}\mathbf{e}_\beta d_3^{\circ\gamma}\mathbf{e}_\gamma d_4^{\circ\delta}\mathbf{e}_\delta$$

which becomes, by taking the 24 permutations of $\alpha \quad \beta \quad \gamma \quad \delta$ and adding

$$\begin{vmatrix} d^{\circ 1}_1 & d^{\circ 1}_2 & d^{\circ 1}_3 & d^{\circ 1}_4 \\ d^{\circ 2}_1 & d^{\circ 2}_2 & d^{\circ 2}_3 & d^{\circ 2}_4 \\ d^{\circ 3}_1 & d^{\circ 3}_2 & d^{\circ 3}_3 & d^{\circ 3}_4 \\ d^{\circ 4}_1 & d^{\circ 4}_2 & d^{\circ 4}_3 & d^{\circ 4}_4 \end{vmatrix} \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$$

or
$$d\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$$

Thus the multiplier d converts the old prime tetraped $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$ into the new one $\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4$.

5. The **c**-lattice has its own set of constants corresponding to the $g_{\alpha\beta} \quad g^{\alpha\beta}$ of the e-lattice. These are the ten constants $h_{\alpha\beta}$ given by

$$h_{\alpha\beta} = \mathbf{c}_\alpha \cdot \mathbf{c}_\beta$$

and the ten $h^{\alpha\beta}$ given by the 16 equations (not all independent)

$$h^{\alpha\rho}h_{\beta\rho} = \delta^\alpha_\beta$$

The expression of $\mathbf{c}_\alpha \cdot \mathbf{c}_\beta$ in terms of the old prime vectors gives

$$h_{\alpha\beta} = d^{\circ\rho}_\alpha \mathbf{e}_\rho \cdot d^{\circ\sigma}_\beta \mathbf{e}_\sigma = g_{\rho\sigma} d^{\circ\rho}_\alpha d^{\circ\sigma}_\beta$$

6. The derived lattice of the **c**-lattice has its prime vectors $\mathbf{c}^\alpha$ given by

$$\mathbf{c}^\alpha = h^{\alpha\beta}\mathbf{c}_\beta \quad \text{or} \quad \mathbf{c}_\alpha = h_{\alpha\beta}\mathbf{c}^\beta$$

And the relation $\mathbf{e}_\alpha = f^{\circ\rho}_\alpha \mathbf{c}_\rho$ has in consequence the equivalent forms

$$\mathbf{e}_\alpha{}' = f_{\alpha\rho}\mathbf{c}^\rho$$

where $f_{\alpha\rho} = h_{\rho\sigma} f^{\circ\sigma}_\alpha$.

We shall also want the quantities $f^{\alpha\rho} \quad f^\alpha_{\circ\rho}$ defined by

$$f^{\alpha\rho} = h^{\alpha\sigma} f^{\circ\rho}_\sigma \qquad f^\alpha_{\circ\rho} = h^{\alpha\sigma} f_{\sigma\rho}$$

Along with the 16 constants $d_{\alpha}^{\circ\beta}$ we shall use $d^{\alpha\beta}$ $d_{\alpha\beta}$ $d^{\alpha}_{\circ\beta}$ defined by

$$d^{\alpha\beta} = g^{\alpha\sigma}d_{\sigma}^{\circ\beta} \qquad d_{\alpha\beta} = g^{\beta\sigma}d_{\alpha}^{\circ\sigma} \qquad d^{\alpha}_{\circ\beta} = g_{\beta\sigma}d^{\alpha\sigma}$$

The second of these shows that $\mathbf{c}_{\alpha} = d_{\alpha}^{\circ\rho}\mathbf{e}_{\rho}$ may be written

$$\mathbf{c}_{\alpha} = d_{\alpha\rho}\mathbf{e}^{\rho}$$

7. The transformation coefficients $d_{\alpha}^{\circ\beta}$ $d_{\alpha\beta}$ $f_{\alpha}^{\cdot\beta}$ $f_{\alpha\beta}$ have simple geometrical meanings. Thus

$$\begin{aligned} d_{\alpha}^{\circ\beta} &= d_{\alpha}^{\circ\rho}\delta_{\rho}^{\beta} = d_{\alpha}^{\circ\rho}\mathbf{e}_{\rho}\cdot\mathbf{e}^{\beta} = \mathbf{c}_{\alpha}\cdot\mathbf{e}^{\beta} \\ d_{\alpha\beta} &= g_{\beta\rho}d_{\alpha}^{\circ\rho} = g_{\beta\rho}\mathbf{c}_{\alpha}\cdot\mathbf{e}^{\rho} = \mathbf{c}_{\alpha}\cdot\mathbf{e}_{\beta} \\ f_{\alpha}^{\circ\beta} &= \mathbf{e}_{\alpha}\cdot\mathbf{c}^{\beta} \\ f_{\alpha\beta} &= \mathbf{e}_{\alpha}\cdot\mathbf{c}_{\beta} \end{aligned}$$

We note that $f_{\alpha\beta} = d_{\beta\alpha}$.

8. Let the vector $\mathbf{x}$ with components x^{α} in the $\mathbf{e}$-lattice be called $\mathbf{s}$ in the $\mathbf{c}$-lattice and have the components s^{α}, so that

$$x^{\alpha}\mathbf{e}_{\alpha} = \mathbf{x} = \mathbf{s} = s^{\alpha}\mathbf{c}_{\alpha}$$

The expression of $\mathbf{c}_{\alpha}$ in terms of $\mathbf{e}_{\alpha}$ gives

$$x^{\rho}\mathbf{e}_{\rho} = s^{\alpha}d_{\alpha}^{\circ\rho}\mathbf{e}_{\rho}$$

so that the old components are given in terms of the new by

$$x^{\rho} = s^{\alpha}d_{\alpha}^{\circ\rho}$$

The expression of $\mathbf{e}_{\alpha}$ in terms of $\mathbf{c}_{\alpha}$ gives similarly

$$s^{\rho} = x^{\alpha}f_{\alpha}^{\circ\rho}$$

The relations between the derived components x_{α} and s_{α} are somewhat more complicated. Thus

$$x_{\sigma} = g_{\sigma\rho}x^{\rho} = g_{\sigma\rho}d_{\alpha}^{\circ\rho}s^{\alpha} = g_{\sigma\rho}d_{\alpha}^{\circ\rho}h^{\alpha\omega}s_{\omega}$$

and

$$s_{\sigma} = h_{\sigma\rho}s^{\rho} = h_{\sigma\rho}x^{\alpha}f_{\alpha}^{\circ\rho} = h_{\sigma\rho}f_{\alpha}^{\circ\rho}g^{\alpha\omega}x_{\omega}$$

9. If two vectors **x** **y** in the **e**-lattice are called **s** **t** in the **c**-lattice, their inner product is

$$g_{\alpha\beta}x^{\alpha}y^{\beta} = g_{\alpha\beta}s^{\theta}d_{\theta}^{\circ\alpha}t^{\phi}d_{\phi}^{\circ\beta}$$

and by the last equation of article 5 this becomes

$$h_{\theta\phi}s^{\theta}t^{\phi}$$

so that the inner product has the same form in terms of the new components and the new lattice constants that it had in the old.

As a particular case, when **y** = **x**, we see that the size-number of a vector, that is the fundamental quadratic expression, has the same form in the new system as in the old.

10. Suppose the vectors **x** **y** **z** **k** **l** **m** to have the components x^{α}, etc., in the **e**-lattice and the components ξ^{α} η^{α} ζ^{α} κ^{α} λ^{α} μ^{α} in the **c**-lattice. So that

$$x^{\alpha} = \xi^{\omega}d_{\omega}^{\circ\alpha}$$

with similar expressions for the other vectors. We will consider the transformation of $\varpi_{\mathbf{xyz},\mathbf{klm}}$ the inner product of the two volume vectors **xyz** **klm** to the **c**-lattice. We have

$$\varpi_{\mathbf{xyz},\mathbf{klm}} = {}^{1}/_{6}\,\overline{xyz}^{\alpha\beta\gamma}\overline{klm}_{\alpha\beta\gamma} = {}^{1}/_{6}\,g_{\alpha\rho}g_{\beta\sigma}g_{\gamma\tau}\overline{xyz}^{\alpha\beta\gamma}\overline{klm}^{\rho\sigma\tau}$$

Now

$$\overline{xyz}^{\alpha\beta\gamma} = \begin{vmatrix} \xi^{\theta}d_{\theta}^{\circ\alpha} & \xi^{\phi}d_{\phi}^{\circ\beta} & \xi^{\psi}d_{\psi}^{\circ\gamma} \\ \eta^{\theta}d_{\theta}^{\circ\alpha} & \eta^{\phi}d_{\phi}^{\circ\beta} & \eta^{\psi}d_{\psi}^{\circ\gamma} \\ \zeta^{\theta}d_{\theta}^{\circ\alpha} & \zeta^{\phi}d_{\phi}^{\circ\beta} & \zeta^{\psi}d_{\psi}^{\circ\gamma} \end{vmatrix} = \overline{\xi\eta\zeta}^{\theta\phi\psi}d_{\theta}^{\circ\alpha}d_{\phi}^{\circ\beta}d_{\psi}^{\circ\gamma}$$

and similarly

$$\overline{klm}^{\rho\sigma\tau} = \overline{\kappa\lambda\mu}^{\epsilon\iota\omega}d_{\epsilon}^{\circ\rho}d_{\iota}^{\circ\sigma}d_{\omega}^{\circ\tau}$$

Also

$$h_{\theta\epsilon} = g_{\alpha\rho}d_{\theta}^{\circ\alpha}d_{\epsilon}^{\circ\rho} \qquad h_{\phi\iota} = g_{\beta\sigma}d_{\phi}^{\circ\beta}d_{\iota}^{\circ\sigma} \qquad h_{\psi\omega} = g_{\gamma\tau}d_{\psi}^{\circ\gamma}d_{\omega}^{\circ\tau}$$

Hence

$$\begin{aligned}\varpi_{\mathbf{xyz,klm}} &= {}^1/6\ \overline{\xi\eta\zeta}^{\theta\phi\psi}\,\overline{\kappa\lambda\mu}^{\iota\iota\omega} h_{\theta\iota} h_{\phi\iota} h_{\psi\omega} \\ &= {}^1/6\ \overline{\xi\eta\zeta}^{\theta\phi\psi}\,\overline{\kappa\lambda\mu}_{\theta\phi\psi}\end{aligned}$$

11. Thus while an element like $\overline{xyz}^{\alpha\beta\gamma}$ of the expression for the inner product has a somewhat complicated expression in terms of the **c**-lattice components, the inner product itself has exactly the same form in terms of the **c**-components as in terms of the **e**-components. For this reason the expression for the inner product is called an invariant.

12. In the same way the inner product of two area vectors is shown to be invariant, and as a size-number is a particular case of an inner product it follows that the size-number of an area vector and of a volume vector are invariant.

13. That outer products are invariant is immediately seen, since for example **xyz** takes the form

$$x^\alpha y^\beta z^\gamma \mathbf{e}_{\alpha\beta\gamma} = {}^1/6\ \overline{xyz}^{\alpha\beta\gamma}\mathbf{e}_{\alpha\beta\gamma}$$

and also the form

$$\xi^\alpha \eta^\beta \zeta^\gamma \mathbf{c}_{\alpha\beta\gamma} = {}^1/6\ \overline{\xi\eta\zeta}^{\alpha\beta\gamma}\mathbf{c}_{\alpha\beta\gamma}$$

The Importance of Invariants

14. Thus the measure of size-numbers, of outer products, and of inner products, is the same in all lattice systems obtained by linear transformation from one in which light travels in straight lines (see chapter 3). Thus these quantities are invariants, and therein lies their merit. For if we express a law in terms of them it is true whether our reference lattice is made up of Greenwich time and the metre stick or whether it is the lattice of some observer in motion relative to the earth.

As an example of the manner in which a quantity in the fourfold takes the place of a quantity in Euclidian space we shall presently discuss the "velocity vector," that is the fourfold vector which specifies the velocity of a point and takes the place of the vector of three-dimensional space.

Einstein's work was done in terms of the invariants of the fourfold. It was this method that made possible that marvellous advance in the laws of motion, which replaced the law that *the path of a planet is an ellipse* by the law that *the world-line of a planet is a geodesic in the fourfold.*

15. We now consider the tensor operator $\mathbf{C} \equiv d_\alpha^{\circ\beta}\mathbf{e}^\alpha\mathbf{e}_\beta$ along with the operator $\mathbf{E} = f_\alpha^{\circ\beta}\mathbf{c}^\alpha\mathbf{c}_\beta$; where $d_\alpha^{\circ\beta}$ $f_\alpha^{\circ\beta}$ are the transformation constants of article 1. **C** may be expressed as $d^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta$ where $d^{\alpha\beta} = g^{\alpha\omega}d_\omega^{\circ\beta}$ and in other ways of the same type. Since $\mathbf{c}_\alpha = h_{\alpha\omega}\mathbf{c}^\omega$ the corresponding expression for **E** is $f^{\alpha\beta}\mathbf{c}_\alpha\mathbf{c}^\beta$ where $f^{\alpha\beta} = h^{\alpha\omega}f_\omega^{\circ\beta}$.

16. Each of these operators carries out a linear transformation of the components of any vector. Such a transformation we have seen to be determined when the transformation of four vectors is given. The forehand stroke of **C** transforms the four vectors $\mathbf{e}_\alpha$ into the four $\mathbf{c}_\alpha$ and may be looked on as determined in this way. **E**'s forehand stroke transforms $\mathbf{c}_\alpha$ to $\mathbf{e}_\alpha$ and is determined by this.

Thus **E** is the reciprocal of **C**, that is to say, it reverses what **C** does, and either operator superposed on the other gives the identity operator.

17. **Exercise.**—By calculating $\mathbf{C} \bullet \mathbf{E}$ from the expressions for **C** and **E** show that $\mathbf{C} \bullet \mathbf{E} = \mathbf{I}$.

18. Let operator **E** act by forehand stroke on the vector $\overrightarrow{OP} \equiv k^\alpha\mathbf{c}_\alpha$ and convert it to $\overrightarrow{OQ}$. Then

$$\overrightarrow{OQ} = k^\alpha\mathbf{c}_\alpha \bullet f_\rho^{\circ\sigma}\mathbf{c}^\rho\mathbf{c}_\sigma = k^\alpha f_\alpha^{\circ\sigma}\mathbf{c}_\sigma = k^\alpha\mathbf{e}_\alpha$$

so that $\overrightarrow{OQ}$ has in the **e**-lattice the same components that $\overrightarrow{OP}$ has in the **c**-lattice.

Let $\overrightarrow{OQ}$ have components p^α in the **c**-lattice, that is let $\overrightarrow{OQ} = p^\alpha \mathbf{c}_\alpha$. We have now

$$p^\alpha d_\alpha^{\circ\rho} \mathbf{e}_\rho = p^\alpha \mathbf{c}_\alpha = k^\alpha \mathbf{e}_\alpha = k^\alpha f_\alpha^{\circ\rho} \mathbf{c}_\rho$$

which gives

$$k^\alpha = d_\omega^{\circ\alpha} p^\omega$$

$$p^\alpha = f_\omega^{\circ\alpha} k^\omega$$

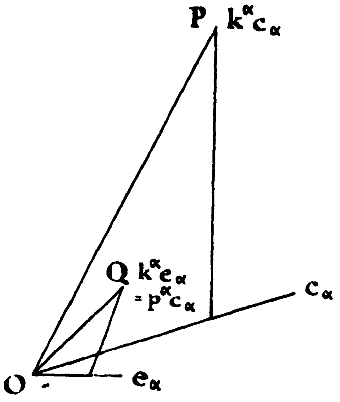


Figure for article 16,18.

19. The size-number of $\overrightarrow{OQ}$ is $g_{\alpha\beta}k^\alpha k^\beta$ while that of $\overrightarrow{OP}$ is $h_{\alpha\beta}k^\alpha k^\beta$. These are in general different, being equal for all vectors only if $h_{\alpha\beta} = g_{\alpha\beta}$.

20. The fact that operator **E** changes the size-number of a vector it acts on is quite consistent with the fact that the transformation of articles 1-7 leaves the size-number unchanged. For that transformation is concerned with the two expressions $k^\alpha \mathbf{e}_\alpha$ $p^\alpha \mathbf{c}_\alpha$ for $\overrightarrow{OQ}$ while the operator converts

to $\overrightarrow{OQ}$ a vector $k^{\alpha}\mathbf{c}_{\alpha}$ formed by using the components of the one lattice with the prime vectors of the other.

21. To see the effect of $\mathbf{E}$ on $\mathbf{c}^{\alpha}$ we make $\overrightarrow{OP}$ coincide with $\mathbf{c}^{\alpha}$ by putting $k^{\omega}\mathbf{c}_{\omega} = h^{\omega\alpha}\mathbf{c}_{\omega}$. Then $h^{\omega\alpha}\mathbf{c}_{\omega} \cdot \mathbf{E}$ is $h^{\omega\alpha}\mathbf{e}_{\omega}$. In general this differs from $\mathbf{e}^{\alpha}$ which is $g^{\omega\alpha}\mathbf{e}_{\omega}$. The two are equal only if

$$h^{\alpha\beta} = g^{\alpha\beta}$$

22. We have the original **e**-lattice with prime vectors $\mathbf{e}_{\alpha}$ and the derived **e**-lattice with prime vectors $\mathbf{e}^{\alpha}$. When we wish to refer to the system of reference made up of the original and derived lattices we shall speak of the **e**-system. Similarly the **c**-system consists of the original and derived **c**-lattices.

In these terms the result of the preceding article is that the operator $\mathbf{E}$ converts the **c**-system into the **e**-system if $h_{\alpha\beta} = g_{\alpha\beta}$.

23. **Exercise.**—Show that the tensor operator $\mathbf{kp} + \mathbf{lq} + \mathbf{mr} + \mathbf{ns}$ cannot convert any finite vector to zero if $\mathbf{k}$ $\mathbf{l}$ $\mathbf{m}$ $\mathbf{n}$ are four independent vectors and $\mathbf{p}$ $\mathbf{q}$ $\mathbf{r}$ $\mathbf{s}$ also four independent vectors.

24. **Exercise.**—What vectors does the operator $\mathbf{kp} + \mathbf{lq} + \mathbf{mr}$ convert to zero by forehand action? and by backhand action?

What vectors does the operator $\mathbf{kp} + \mathbf{lq}$ convert to zero by each action?

What vectors can the operator $\mathbf{kp}$ convert to zero?

CHAPTER 17

THE CONGRUENCE OPERATOR

1. We take two lattices, the **c**-lattice with prime vectors $\mathbf{c}_\alpha$ and lattice constants $h_{\alpha\beta}$, and the **e**-lattice with prime vectors $\mathbf{e}_\alpha$ and lattice constants $g_{\alpha\beta}$. We suppose the ten lattice constants to be the same for both lattices, that is $h_{\alpha\beta}=g_{\alpha\beta}$. We take the tensor operator

$$\mathbf{R} = \mathbf{c}_\alpha \mathbf{e}^\alpha$$

and suppose that by backhand action it converts $\mathbf{x} = x^\alpha \mathbf{e}_\alpha$ to $\mathbf{z}$, so that

$$\mathbf{z} = \mathbf{R} \cdot \mathbf{x} = \mathbf{c}_\alpha(\mathbf{e}^\alpha \cdot \mathbf{x}) = \mathbf{c}_\alpha x^\alpha$$

and $\mathbf{z}$ has the same components in the $\mathbf{c}_\alpha$-lattice that $\mathbf{x}$ has in the $\mathbf{e}_\alpha$-lattice.

2. Hence

$$\nu_{\mathbf{z}} = h_{\alpha\beta} x^\alpha x^\beta = g_{\alpha\beta} x^\alpha x^\beta = \nu_{\mathbf{x}}$$

and if **R** converts **y** into **u**

$$\varpi_{\mathbf{zu}} = \mathbf{c}_\alpha x^\alpha \cdot \mathbf{c}_\beta y^\beta = h_{\alpha\beta} x^\alpha y^\beta$$
$$= g_{\alpha\beta} x^\alpha y^\beta = \varpi_{\mathbf{xy}}$$

We denote the heads of vectors **x** **y** **z** **u** by X Y Z U and have

$$\begin{aligned}\nu_{XY} &= g_{\alpha\beta}(y^\alpha - x^\alpha)(y^\beta - x^\beta)\\ &= g_{\alpha\beta}(y^\alpha y^\beta + x^\alpha x^\beta - y^\alpha x^\beta - x^\alpha y^\beta)\\ &= \nu_{\mathbf{x}} + \nu_{\mathbf{y}} - 2\varpi_{\mathbf{xy}}\end{aligned}$$

and a similar expression for ν_{ZU}. Therefore $\nu_{ZU} = \nu_{XY}$ and the operator **R** leaves the size-number of every separation unchanged. In other words **R** converts any triangle into a congruent triangle.

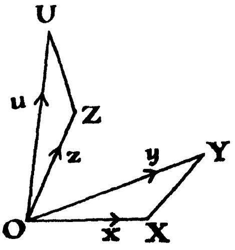


Figure for article 17,2.

In particular **R** converts the four prime vectors $\mathbf{e}_\alpha$ into the four prime vectors $\mathbf{c}_\alpha$, and the two five-faces formed by the sets of prime vectors are congruent.

3. **Exercise.**—Show that the two figures, one consisting of the 8 prime vectors of the **c**-system and the other of the 8 prime vectors of the **e**-system, are congruent.

4. We observe that the operator conjugate to **R** is $\overline{\mathbf{R}} \equiv \mathbf{e}^\alpha \mathbf{c}_\alpha$ so that

$$\mathbf{R} \cdot \overline{\mathbf{R}} = \mathbf{c}_\alpha \mathbf{e}^\alpha \cdot \mathbf{e}^\beta \mathbf{c}_\beta = g^{\alpha\beta} \mathbf{c}_\alpha \mathbf{c}_\beta = \mathbf{c}_\alpha \mathbf{c}^\alpha$$
$$\overline{\mathbf{R}} \cdot \mathbf{R} = \mathbf{e}^\alpha \mathbf{c}_\alpha \cdot \mathbf{c}_\beta \mathbf{e}^\beta = g_{\alpha\beta} \mathbf{e}^\alpha \mathbf{e}^\beta = \mathbf{e}^\alpha \mathbf{e}_\alpha$$

confirming our knowledge that $\mathbf{R} \cdot \overline{\mathbf{R}}$ and $\overline{\mathbf{R}} \cdot \mathbf{R}$ must both be the identity operator.

5. We use the term congruence operator for any operator that leaves the size-numbers of all separations unchanged. We shall see that in some cases it is possible to bring two congruent figures to coincidence by a continuous motion,* while in others a discontinuous change analogous to reflexion in Euclidian space is involved. When the two figures have a pair of corresponding points coincident the continuous motion that brings about complete coincidence is called rotation.

Since a tensor operator has 16 constants, and by requiring the **c**-lattice to have the constants of the **e**-lattice we impose 10 conditions a congruence operator has 6 constants at disposal.

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* In the fourfold conception of the world there is no motion ; corresponding to the classical conception of a particle in motion we have an unchanging line of point-events. When we speak of bringing two congruent figures to coincidence, we mean that the two can be connected by an infinite series of figures congruent to them such that any two successive figures differ only infinitesimally in position.

CHAPTER 18

REFLEXION

1. THERE are four kinds of reflexion. The mirror may be a triplane, a biplane, a line, or a point. The line joining a point and its image must meet the mirror, must be bisected by it, and except in the case of the point-mirror must be perpendicular to it. This is the definition of reflexion in the fourfold.

2. We take the triplane $\mathbf{e}_{123}$ as mirror. By article 15,1 the projector of a vector $\mathbf{x}$ is $x^4\mathbf{e}^4/g^{44}$, so that the image of $\mathbf{x}$ is

$$\mathbf{x} - 2x^4\mathbf{e}^4/g^{44}$$

and the operator $\mathbf{Q}_{123}$ which produces this reflexion is *

$$\mathbf{Q}_{123} = \mathbf{I} - 2\mathbf{e}^4\mathbf{e}^4/g^{44}$$

3. **Exercise.**—Verify that $\mathbf{Q}_{123}$ leaves every vector in the triplane $\mathbf{e}_{123}$ unmoved, and find the images of $\mathbf{e}_4$ and $\mathbf{e}^4$.

4. For a biplane mirror we take $\mathbf{e}_{12}$. The operator that converts a vector $\mathbf{x}$ into its projection is, by article 14,3

$$\mathbf{K} \equiv - \mathbf{P} \bullet \mathbf{P}/g_{1212}$$

where $\mathbf{P} = \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1$
while the operator $\mathbf{I} + \mathbf{P} \bullet \mathbf{P}/g_{1212}$ converts $\mathbf{x}$ into its projector.

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* Here and throughout the chapter the operator is supposed to use the back-hand stroke.

Since $\mathbf{P} = (g_{1\theta}g_{2\phi} - g_{2\theta}g_{1\phi})\mathbf{e}^\theta\mathbf{e}^\phi = g_{12\theta\phi}\mathbf{e}^\theta\mathbf{e}^\phi$ the projection operator is

$$-(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1)\bullet g_{12\theta\phi}\mathbf{e}^\theta\mathbf{e}^\psi/g_{1212} = -(\mathbf{e}_1 g_{122\phi} - \mathbf{e}_2 g_{121\phi})\mathbf{e}^\phi/g_{1212}$$

and the projector operator

$$\begin{aligned}&(\mathbf{e}_\phi g_{1212} + \mathbf{e}_1 g_{122\phi} - \mathbf{e}_2 g_{121\phi})\mathbf{e}^\phi/g_{1212}\\ &= (g_{\phi\psi}g_{1212} + g_{1\psi}g_{122\phi} - g_{2\psi}g_{121\phi})\mathbf{e}^\psi\mathbf{e}^\phi/g_{1212}\\ &= \mathbf{e}^\phi\mathbf{e}^\psi g_{\psi12\phi12}/g_{1212}\end{aligned}$$

Now $g_{\psi12\phi12}$ vanishes if either ϕ or ψ is 1 or 2, so that by the help of article 4,10 we may write the operator as

$$\Sigma\mathbf{e}^\lambda\mathbf{e}^\rho g g^{\mu\sigma}/g_{1212}$$

the summation to be carried out for $\lambda = 3$ $\mu = 4$, $\lambda = 4$ $\mu = 3$, $\rho = 3$ $\sigma = 4$, $\rho = 4$ $\sigma = 3$. Hence since by article 12,12 $g_{1212} = gg^{3434}$ the projector operator is

$$\mathbf{L} \equiv (g^{44}\mathbf{e}^3\mathbf{e}^3 - g^{43}\mathbf{e}^3\mathbf{e}^4 - g^{34}\mathbf{e}^4\mathbf{e}^3 + g^{33}\mathbf{e}^4\mathbf{e}^4)/g^{3434}$$

5. If $\mathbf{s}$ is the projection and $\mathbf{p}$ the projector of $\mathbf{x}$ the image of $\mathbf{x}$ is equal to $2\mathbf{s} - \mathbf{x}$ and to $\mathbf{x} - 2\mathbf{p}$. Thus $2\mathbf{K} - \mathbf{I}$ and $\mathbf{I} - 2\mathbf{L}$ are two expressions for $\mathbf{Q}_{12}$, the reflexion operator for the biplane $\mathbf{e}_{12}$; that is to say

$$\begin{aligned}\mathbf{Q}_{12} &= 2(g_{22}\mathbf{e}_1\mathbf{e}_1 - g_{12}\mathbf{e}_2\mathbf{e}_1 - g_{21}\mathbf{e}_1\mathbf{e}_2 + g_{11}\mathbf{e}_2\mathbf{e}_2)/g_{1212} - \mathbf{I}\\ &= \mathbf{I} - 2(g^{44}\mathbf{e}^3\mathbf{e}^3 - g^{34}\mathbf{e}^4\mathbf{e}^3 - g^{43}\mathbf{e}^3\mathbf{e}^4 + g^{33}\mathbf{e}^4\mathbf{e}^4)/g^{3434}\end{aligned}$$

6. For a straight line mirror we take $\mathbf{e}^4$. By article 13,18 the operator $\mathbf{e}^4\mathbf{e}^4/g^{44}$ converts any vector into its projection. Hence the operator

$$\mathbf{Q}^4 \equiv 2\mathbf{e}^4\mathbf{e}^4/g^{44} - \mathbf{I}$$

converts it into its image.

7. As point-mirror we take the origin. The operator $\mathbf{Q}_0$ which causes the reflexion is

$$\mathbf{Q}_0 = -\mathbf{e}_\alpha\mathbf{e}^\alpha = -\mathbf{I}$$

8. In Euclidian space reflexion in a line is equivalent to a pure rotation, while for reflexion in a plane or in a point there is no equivalent pure rotation. We shall discuss the corresponding properties in the fourfold.

9. We consider the **e**-lattice and its image in the triplane $\mathbf{e}_{123}$, which we call the **f**-lattice. The first three prime vectors of the **f**-lattice coincide with the first three of the **e**-lattice. We denote the head of each vector by the corresponding capital letter, so that for example E_1 is the head of $\mathbf{e}_1$.

The constants $g_{\alpha\beta}$ being supposed given and the position of $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ being chosen the head of the fourth vector $\mathbf{e}_4$ or $\mathbf{f}_4$ is given by

$$\mathbf{x}\cdot\mathbf{e}_1 = g_{41} \qquad \mathbf{x}\cdot\mathbf{e}_2 = g_{42} \qquad \mathbf{x}\cdot\mathbf{e}_3 = g_{43} \qquad \mathbf{x}\cdot\mathbf{x} = g_{44}$$

that is it is one of the two intersections of the contour trisurface of size-number g_{44} by the straight line given by the first three equations. E_4 and F_4 satisfy these conditions and are the two points in question. No other position being possible for the head of the fourth vector it is impossible for lattice **f** to come by continuous movement to coincidence with lattice **e**.

10. As biplane mirror we take $\mathbf{e}^{34}$, and the image of the **e**-lattice in it we call the **h**-lattice. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular to the mirror, $\mathbf{h}_1$ is $\mathbf{e}_1$ reversed and $\mathbf{h}_2$ is $\mathbf{e}_2$ reversed. If we hold $\mathbf{h}^3$ $\mathbf{h}^4$ still, in coincidence with $\mathbf{e}^3$ $\mathbf{e}^4$, then $\mathbf{h}_1$ $\mathbf{h}_2$ being perpendicular to $\mathbf{h}^3$ $\mathbf{h}^4$ are limited to the biplane $\mathbf{e}_{12}$ and what we have to consider is rotation in that biplane.

11. If $\mathbf{e}_{12}$ is a mixed biplane it cuts the contour system in a series of hyperbolas. Hence the head X of any vector

$\overrightarrow{OX}$ is limited to one branch of its hyperbola and cannot be brought by continuous motion to the reverse position.

12. Another observation is worth making about rotation in a mixed biplane. Let the vectors $\overrightarrow{OX}$ $\overrightarrow{OY}$ be perpendicular to one another. Let $\overrightarrow{OX}$ rotate towards P in the figure. Then since perpendicularity is conjugacy with respect to the hyperbolas $\overrightarrow{OY}$ will rotate towards Q. That is to say the effect of rotation is to turn $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in what the Euclidian would call opposite directions.

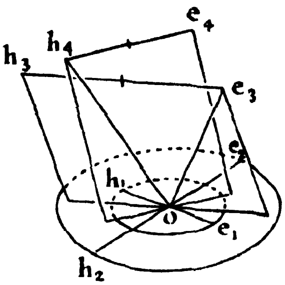
13. If $\mathbf{e}_{12}$ is a spacelike biplane it cuts every contour in an ellipse. H_1 lies on the same ellipse as E_1 and can travel along the ellipse to coincidence. As the inclination between $\mathbf{h}_1$ $\mathbf{h}_2$ remains constant the rotation that brings H_1 to E_1 also brings H_2 to E_2; and for the same reason it reverses every vector in the biplane.

Take any vector $\overrightarrow{OZ}$. Through Z draw a biplane parallel to $\mathbf{e}_{12}$ and meeting $\mathbf{e}^{34}$ in Y. Take M as the fourth corner of the parallelogram $OYZM$. Then OM lies in $\mathbf{e}_{12}$ and ZM is parallel to $\mathbf{e}^{34}$. Take L as the point to which the rotation brings M, so that O is the mid point of LM. Take X as the fourth corner of the parallelogram $YOLX$.

The rotation brings M to L, leaves OY unmoved, and a parallelogram remains a parallelogram. Therefore Z comes to X. And since ZX is bisected at Y at right angles by $\mathbf{e}^{34}$, every point rotates to its image.

Hence any figure can be brought by a pure rotation to coincidence with its image in a mixed biplane, but cannot be brought by such means to coincidence with its image in a spacelike biplane.

14. We observe that if $\mathbf{h}_1$ $\mathbf{e}^3$ $\mathbf{e}^4$ form a right-handed set of three directions, the set $\mathbf{e}_1$ $\mathbf{e}^3$ $\mathbf{e}^4$ to coincidence with which they come by rotation are a left-handed set. Thus the sharp distinction in Euclidian space between right- and left-handedness does not hold in the fourfold in which it is sometimes possible to rotate right- and left-handed sets to coincidence.



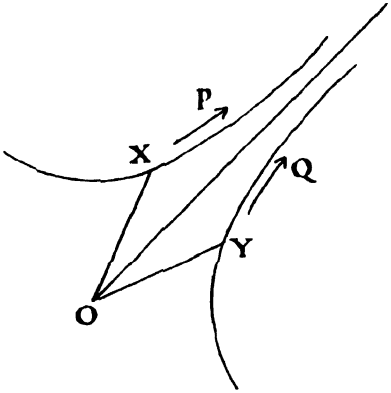


Figure for 18,12.

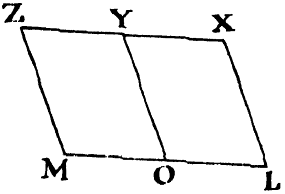


Figure for 18,13.

15. We may look on this rotation in the biplane $\mathbf{e}_{12}$ as taking place in any triplane that contains $\mathbf{e}_{12}$, for example in the triplane $\mathbf{e}_{123}$. The axis about which the rotation takes place in $\mathbf{e}_{123}$ must be perpendicular to $\mathbf{e}_{12}$ and it must lie in $\mathbf{e}_{123}$, that is it is perpendicular to $\mathbf{e}^4$. This is (by article 13,12) the position to which operator $\mathbf{e}^3\mathbf{e}^4 - \mathbf{e}^4\mathbf{e}^3$ throws $\mathbf{e}^4$, so that the axis is $g^{44}\mathbf{e}^3 - g^{43}\mathbf{e}^4$.

More generally we may take as axis any vector $\mathbf{r}$ perpendicular to $\mathbf{e}_{12}$, the triplane of the rotation being that of $\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{r}$.

16. Let the $\mathbf{k}$-lattice be the image in $\mathbf{e}^4$ of the $\mathbf{e}$-lattice. As $\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3$ are perpendicular to $\mathbf{e}^4$ reflexion reverses them. A timelike vector cannot be reversed by continuous motion, for reversal means moving it from one cup of its trihyperboloid to the other. Thus for the $\mathbf{k}$-lattice to come by continuous motion to coincidence one condition is that in the $\mathbf{e}$-lattice $\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3$ must be spacelike.

17. We assume the triplane $\mathbf{e}_{123}$ to be spacelike, so that

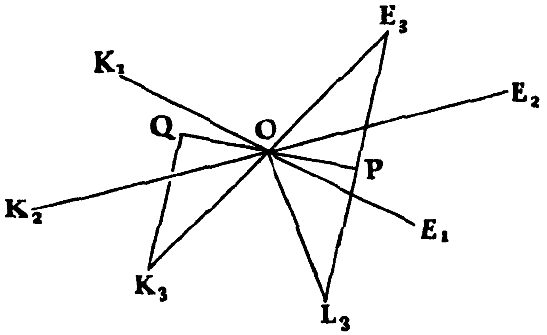


Figure for article 18,17.

the vector $\mathbf{e}^4$ is timelike, and the biplane $\mathbf{e}^{34}$ is of mixed character. We carry out the pure rotation in the biplane $\mathbf{e}_{12}$ which brings $\mathbf{k}_1\ \mathbf{k}_2$ to coincidence with $\mathbf{e}_1\ \mathbf{e}_2$. This position to which the $\mathbf{k}$-lattice is thus brought we call the $\mathbf{l}$-lattice.

We let fall perpendiculars $E_3P\ K_3Q$ from $E_3\ K_3$ upon $\mathbf{e}_{12}$. E_3OK_3 being a straight line bisected at O, so also is POQ. And $E_3P = K_3Q$. The rotation brings OQ to OP and leaves QK_3, the other component of OK_3, unaltered. So that K_3 comes to L_3 such that PL_3 is equal and parallel to QK_3 and to E_3P. That is to say K_3 comes to the image of E_3 in $\mathbf{e}_{12}$.

18. To find L_4 the point to which K_4 comes we let fall perpendiculars E_4X K_4U from E_4 and K'_4 on the triplane $\mathbf{e}_{123}$. Since E_4K_4 is normal to $\mathbf{e}^4$ and bisected by $\mathbf{e}^4$ at K' it is parallel to $\mathbf{e}_{123}$, so that the two perpendiculars are equal, and XOU is a straight line bisected at O.

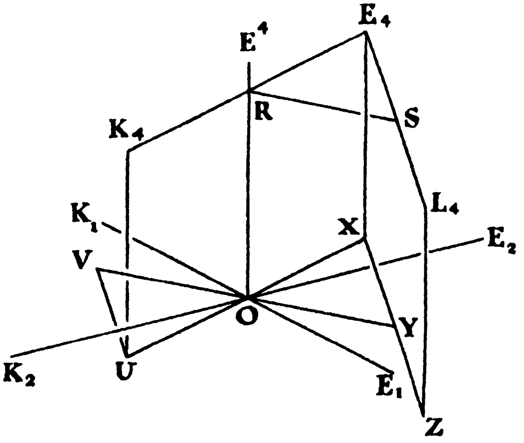


Figure for article 18,18.

Now from X U we let fall perpendiculars XY UV on e_{12}. They are equal and YOV is a straight line bisected at O. The rotation swings OV to OY, and translates VU UK_4 without change of size or direction since they are normal to $\mathbf{e}_{12}$. U comes to Z the image of X in $\mathbf{e}_{12}$ and K_4 comes to L_4 the fourth corner of the parallelogram E_4XZL_4.

RS drawn parallel to OY meets E_4L_4 in its middle point S, and E_4L_4 is parallel to XZ and perpendicular to $\mathbf{e}_{12}$.

Thus K_4 comes to the image of E_4 in the biplane through R parallel to $\mathbf{e}_{12}$.

19. The line E_3L_3 of article 17 meets the triplane $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$ in one point. That point is P which lies in $\mathbf{e}_{12}$ and therefore in the triplane. E_3L_3 is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$, and since it lies in $\mathbf{e}_{123}$ it is perpendicular to $\mathbf{e}^4$. It is thus perpendicular to the triplane $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$, and L_3 is the image of E_3 in $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$.

The line E_4L_4 of article 18 meets $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$ in one point. Since RS meets $\mathbf{e}^4$ and is parallel to $\mathbf{e}_{12}$ S is in $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$ and is the point. E_4L_4 is normal to $\mathbf{e}_{12}$ and parallel to $\mathbf{e}_{123}$, and is therefore normal to $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$.

20. The **l**-lattice is thus the image in the triplane $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^4$ of the **e**-lattice, and by article 8 cannot be brought by a pure rotation to coincidence with it.

Therefore the **k**-lattice cannot by pure rotation be brought to coincidence with the **e**-lattice. There is thus no pure rotation equivalent to reflexion in a line.

21. For a point-mirror we take the origin. Then reflexion reverses every vector at the origin, including the timelike vectors. As a timelike vector cannot be reversed by continuous motion there is no pure rotation equivalent to reflexion by a point-mirror.

22. This discussion of reflexion shows that in the fourfold there are a great variety of cases in which congruent figures cannot be brought to coincidence by rotation.

CHAPTER 19

THE VELOCITY VECTOR

1. IN the classical conception of the world the history of a particle was given by expressing three rectangular space coordinates x y z in terms of the time t. In the relativity conception we substitute a fourfold in which the place of a set of pace coordinates along with the appropriate value of t is taken by a point or point-event. Whereas in the classical conception the history was given by a line in space with a value of t attached to every point of it, now in the relativity conception the history is given by a line in the fourfold each point of which has the four coordinates x y z t. This line is called the world line of the particle. The origin is located in time as well as space, and may be said to be "here and now."

2. Consider a particle whose world line is a straight line through the origin O. Let the coordinates of a point P upon it be x y z t. In classical language we say that it has travelled distances x y z parallel to the three axes in time t, or that its velocity v has components x/t y/t z/t, and that the magnitude of the velocity is

$$v = \sqrt{(x^2 + y^2 + z^2)/t^2}$$

3. The size-number of the separation OP is $t^2 - x^2 - y^2 - z^2$ or $t^2(1 - v^2)$. If OP lies inside the basic tricone the

size-number is positive and v is less than 1. If OP lies outside the size-number is negative and v is greater than 1. If OP is a generator of the tricone v is equal to 1.

Now it is a principle of relativity that no velocity can be greater than that of light, which we have taken as 1. Therefore the world line OP lies on the tricone in the case of light and inside the tricone for anything that moves slower.

4. The size-number $\nu_{OP} \equiv t^2 - x^2 - y^2 - z^2$ and the size $\sigma_{OP} \equiv \sqrt{t^2 - x^2 - y^2 - z^2}$ have the property that their values are the same in all reference lattices; they are invariants. The special symbol s is used for the size, so that

$$s = \sigma_{OP} = \sqrt{t^2 - x^2 - y^2 - z^2} \quad s^2 = \nu_{OP} = t^2 - x^2 - y^2 - z^2.$$

5. We introduce another lattice in which the coordinates of a point are $\xi \quad \eta \quad \zeta \quad \tau$ and the size-number of the separation of that point from the origin $\tau^2 - \xi^2 - \eta^2 - \zeta^2$, the lattice so taken that P lies on the τ-axis. For P and for every other point of our straight world line $\xi \quad \eta \quad \zeta$ are zero, so that the size of OP is τ. Thus the invariant quantity s is the time of this particular reference lattice in which the particle lies at rest.

It is appropriate that the particle should imagine itself to be at rest and everything else in motion, so that for him τ is a unique measure of time. Accordingly the invariant s, which is equal to τ, is called the "proper time" of the particle, the zero of time being at the origin O, here-and-now.

6. If we divide the coordinates of P by s we shall have

$$\frac{x}{s} \quad \frac{y}{s} \quad \frac{z}{s} \quad \frac{t}{s}$$

the coordinates of a point Q on OP. Also x/s is the distance traversed parallel to the x-axis, measured in the $xyzt$-lattice, that elapses per unit of proper time. And so for y/s z/s. And t/s is the amount of time measured in the $xyzt$-lattice, that elapses per unit of proper time. It is accordingly appropriate to call the vector $\overrightarrow{OQ}$ the "velocity vector" of the motion.

7. We observe that the components of the velocity vector and the components of the classical velocity v are related by

$$\frac{x}{s} = \frac{1}{\sqrt{1 - v^2}} \frac{x}{t} \qquad \frac{y}{s} = \frac{1}{\sqrt{1 - v^2}} \frac{y}{t}$$

$$\frac{z}{s} = \frac{1}{\sqrt{1 - v^2}} \frac{z}{t} \qquad \frac{t}{s} = \frac{1}{\sqrt{1 - v^2}}$$

We observe also that the size-number of OQ and of every velocity vector is unity since $s^2 = t^2 - x^2 - y^2 - z^2$.

8. Take now a world line of any form. Let P with coordinates x y z t and Q with coordinates $x+dx$ $y+dy$ $z+dz$ $t+dt$ be two successive points on it. The vector $\overrightarrow{PQ}$ has the components dx dy dz dt and the interval of proper-time between P and Q is

$$ds = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

The components of the velocity vector **u** of the motion at P are

$$\frac{dx}{ds} \qquad \frac{dy}{ds} \qquad \frac{dz}{ds} \qquad \frac{dt}{ds}$$

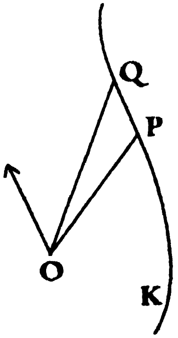


Figure for article 19,8.

the components of the classical velocity v

$$\frac{dx}{dt} \qquad \frac{dy}{dt} \qquad \frac{dz}{dt}$$

and the relation between ds dt is

$$ds = dt\sqrt{1 - v^2}.$$

9. Let us take the zero of proper time for the world line at the point-event K. We divide the piece of the line between K and P into infinitesimal elements, calculate the interval of proper time for each element, and define the proper time at P as the sum of these intervals. That is to say the proper time s at P is

$$s = \int_K^P \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

10. If instead of the cubical (or supercubical) lattice we employ an oblique lattice with prime vectors $\mathbf{e}_\alpha$ and lattice constants $g_{\alpha\beta}$ we denote the vector $\overrightarrow{OP}$ by $\mathbf{x}$ with components x^1 x^2 x^3 x^4 and vector $\overrightarrow{PQ}$ by $d\mathbf{x}$ with components dx^1 dx^2 dx^3 dx^4. The vector $\overrightarrow{OQ}$ is therefore $\mathbf{x} + d\mathbf{x}$ and has components $x^1 + dx^1$ $x^2 + dx^2$ $x^3 + dx^3$ $x^4 + dx^4$.

The interval of proper time from P to Q, or the size of PQ is

$$ds = \sigma_{PQ} = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$$

and zero of proper time on the line being at K, the proper time at P is

$$s = \int_K^P \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$$

11. The components of the velocity vector $\mathbf{u}$ are

$$\frac{dx^1}{ds} \quad \frac{dx^2}{ds} \quad \frac{dx^3}{ds} \quad \frac{dx^4}{ds}$$

and the velocity vector itself is

$$\mathbf{u} = \frac{dx^1}{ds}\mathbf{e}_1 + \frac{dx^2}{ds}\mathbf{e}_2 + \frac{dx^3}{ds}\mathbf{e}_3 + \frac{dx^4}{ds}\mathbf{e}_4$$

which may also be written as

$$\mathbf{u} = \frac{d}{ds}(x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + x^4\mathbf{e}_4)$$

or as

$$\mathbf{u} = \frac{d\mathbf{x}}{ds}$$

12. Article 11 may be stated more in terms of vectors. We start with the vector $\overrightarrow{OP}$ or $\mathbf{x}$ or $x^a\mathbf{e}_a$ with components $x^1 \; x^2 \; x^3 \; x^4$. We add to it the vector $\overrightarrow{PQ}$ or $d\mathbf{x}$ or $dx^a\mathbf{e}_a$ with the four components $dx^1 \; dx^2 \; dx^3 \; dx^4$. The sum of the two is the vector $\overrightarrow{OQ}$ or $\mathbf{x} + d\mathbf{x}$ or $(x^a + dx^a)\mathbf{e}_a$ with the components $x^1 + dx^1 \quad x^2 + dx^2 \quad x^3 + dx^3 \quad x^4 + dx^4$.

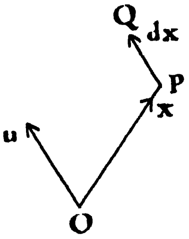


Figure for article 19,12.

The velocity vector $\mathbf{u} = u^a\mathbf{e}_a$ is a scalar multiple of the vector $d\mathbf{x}$ namely $d\mathbf{x}/ds = dx^a\mathbf{e}_a/ds$, so that its components are

$$u^a = dx^a/ds$$

or written in full

$$u^1 = \frac{dx^1}{ds} \qquad u^2 = \frac{dx^2}{ds} \qquad u^3 = \frac{dx^3}{ds} \qquad u^4 = \frac{dx^4}{ds}$$

13. When we wish to split up the fourfold into time and space, the prime vector we choose for the measurement of time must be normal to the triplane which is to be our space. Now $\mathbf{e}_4$ is normal to $\mathbf{e}_1 \; \mathbf{e}_2 \; \mathbf{e}_3$ if $g_{14} \; g_{24} \; g_{34}$ vanish. Accordingly we make these particular $g_{\alpha\beta}$ vanish, take $\mathbf{e}_4$ for time vector and the triplane $\mathbf{e}_{123}$ for our space.

14. The world having been split into space and time the event P is characterized as happening at local time x^4 after

the origin, which is here-and-now; or if we wish to emphasize the unit $\mathbf{e}_4$, as happening at time $x^4\sqrt{g_{44}}$. The place at which it happens has in the $\mathbf{e}_{123}$-lattice the coordinates x^1 x^2 x^3, or if we emphasize the units, the coordinates $x^1\sqrt{-g_{11}}$ $x^2\sqrt{-g_{22}}$ $x^3\sqrt{-g_{33}}$.

"Local time" means the time in this particular reference lattice in contradistinction to proper time.

15. With the world thus split the interval of local time between P and Q is dx^4. The position of Q relative to P has the components dx^1 dx^2 dx^3 so that the distance δ_{PQ} between P and Q is given in our space lattice $\mathbf{e}_{123}$ by

$$\delta^2_{PQ} = -g_{11}dx^1dx^1 - g_{22}dx^2dx^2 - g_{33}dx^3dx^3$$
$$- 2g_{23}dx^2dx^3 - 2g_{31}dx^3dx^1 - 2g_{12}dx^1dx^2$$

which we may write concisely as

$$\delta^2_{PQ} = -\sum_{123} g_{\alpha\beta}dx^\alpha dx^\beta$$

We also write τ_{PQ} to denote $\sqrt{g_{44}}dx^4$ the interval of loca time from P to Q.

The magnitude of the classical velocity at P is

$$v = \delta_{PQ}/\tau_{PQ}$$

16. With the world thus split the interval ds of prope time from P to Q is given by

$$ds^2 = \nu_{PQ} = g_{\alpha\beta}dx^\alpha dx^\beta$$

In this there are ten terms. One is $g_{44}dx^4dx^4$. Six are th terms of δ^2_{PQ}. The remaining three are those in g_{14} g_{24} g and vanish. Therefore

$$ds^2 = \tau^2_{PQ} - \delta^2_{PQ}$$

and

$$ds = \sqrt{(1 - v^2)}\tau_{PQ}$$

The relation between the velocity vector and the classical velocity is accordingly

$$\frac{dx^1}{ds} = \frac{1}{\sqrt{1 - v^2}} \frac{dx^1}{\tau_{PQ}} \qquad \frac{dx^2}{ds} = \frac{1}{\sqrt{1 - v^2}} \frac{dx^2}{\tau_{PQ}}$$

$$\frac{dx^3}{ds} = \frac{1}{\sqrt{1 - v^2}} \frac{dx^3}{\tau_{PQ}} \qquad \frac{dx^4}{ds} = \frac{1}{\sqrt{1 - v^2}} \frac{1}{\sqrt{g_{44}}}$$

CHAPTER 20

THE ROTATION OPERATOR

1. OUR main object in discussing the transformation from one reference lattice to another is the comparison of the pictures of the world as seen by two observers *C* and *E* who are in relative motion. We can interpolate between *C* and *E* as many other observers as we please. We can therefore pass from the *C* lattice by stages as small as we please to the *E* lattice, that is to say by pure rotation. Thus the congruence operators with which we are chiefly concerned are those that cause rotation; we call them rotation operators.

2. Suppose we have two observers, *E* who uses the **e**-lattice, and *C* who uses the **c**-lattice, both having their origin here-and-now at *O*, the two lattices being congruent. Suppose the world line of *E* to be the prime vector $\mathbf{e}_4$; $\mathbf{e}_4$ is therefore timelike, and *E* is at rest in the classical sense in his own reference system. Also suppose *C*'s world line to be $\mathbf{c}_4$.

Consider the operator $\mathbf{R} \equiv \mathbf{c}_\omega \mathbf{e}^\omega$. If, proceeding as in chapter 16 and remembering that $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are equal in the present case, we express the prime vectors of each lattice in terms of the other lattice by

$$\mathbf{c}_\alpha = d_\alpha^{\circ\beta} \mathbf{e}_\beta \qquad \mathbf{e}_\alpha = f_\alpha^{\circ\beta} \mathbf{c}_\beta$$

R takes the forms

$$\mathbf{R} = d_\alpha^{\circ\beta} \mathbf{e}_\beta \mathbf{e}^\alpha = f^{\alpha\beta} \mathbf{c}_\alpha \mathbf{c}_\beta$$

Suppose vector $\overrightarrow{OP}$ expressed in the two lattices to be

$$x^{\alpha}\mathbf{e}_{\alpha} \text{ and } z^{\alpha}\mathbf{c}_{\alpha}$$

Then

$\overrightarrow{OP} = x^{\omega}\mathbf{e}_{\omega} = z^{\omega}\mathbf{c}_{\omega} = x^{\alpha}f_{\alpha}^{\circ\omega}\mathbf{c}_{\omega} = z^{\alpha}d_{\alpha}^{\circ\omega}\mathbf{e}_{\omega}$ for all values of $\overrightarrow{OP}$ so that

$$x^{\omega} = z^{\alpha}d_{\alpha}^{\circ\omega} \qquad z^{\omega} = x^{\alpha}f_{\alpha}^{\circ\omega}$$

3. Suppose that C's velocity vector in the **e**-lattice is **u**. In the **c**-lattice it is $\mathbf{c}_4/\sqrt{g_{44}}$ since it lies along $\mathbf{c}_4$ and has size-number unity. We take this vector for $\overrightarrow{OP}$ and have

$$z^1 = z^2 = z^3 = 0 \qquad z^4 = 1/\sqrt{g_{44}}$$
$$u^{\omega} = x^{\omega} = z^4 d_4^{\circ\omega} = d_4^{\circ\omega}/\sqrt{g_{44}}$$

We have thus four equations for $d_4^{\circ 1}$ $d_4^{\circ 2}$ $d_4^{\circ 3}$ $d_4^{\circ 4}$ which must be satisfied if a prescribed velocity vector **u** is to lie along $\mathbf{c}_4$. The fact that the size-number of **u** is unity gives one relation among these quantities, namely

$$g_{\rho\sigma}d_4^{\circ\rho}d_4^{\circ\sigma} = g_{44}$$

so that the four equations amount to only three independent conditions.

Since a congruence operator has in general six disposable constants, a congruence operator $\mathbf{c}_{\alpha}\mathbf{e}^{\alpha}$ which throws a prescribed velocity vector along $\mathbf{c}_4$ (or rather which converts the **e**-lattice into one which has $\mathbf{e}_4$ in a prescribed direction) has three disposable constants.

4. **Exercise.** — Discuss the possibility of deducing $d_4^{\circ 4} = u^4\sqrt{g_{44}}$ from the other three equations of article 3 and $g_{\rho\sigma}d_4^{\circ\rho}d_4^{\circ\sigma} = g_{44}$.

5. The velocity vectors of E and C head towards the future so that their heads both lie on the future-time bowl of

the contour trisurface of size-number unity. We can therefore by continuous motion bring C_4 to coincidence with E_4. This being done and the values of $g_{\alpha\beta}$ being supposed to be specified C_1 may lie anywhere on the locus

$$\mathbf{x} \bullet \mathbf{e}_4 = g_{14} \qquad \mathbf{x} \bullet \mathbf{x} = g_{11}$$

a conicoid which also contains E_1.

As $\mathbf{e}_4$ is timelike the triplane $\mathbf{x} \bullet \mathbf{e}_4 = g_{14}$ is spacelike and cuts every contour in an ellipsoid. As C_1 lies on the same ellipsoid as E_1 it can be brought by continuous motion to coincidence with it.

When C_4 C_1 coincide with E_4 E_1 the locus of possible positions for C_2 is

$$\mathbf{x} \bullet \mathbf{e}_4 = g_{24} \qquad \mathbf{x} \bullet \mathbf{e}_1 = g_{21} \qquad \mathbf{x} \bullet \mathbf{x} = g_{22}$$

which is the line of intersection of a contour trisurface by a spacelike biplane, and therefore an ellipse. As C_2 lies on the same ellipse as E_2 it can by continuous motion come to coincidence with it.

When C_4 C_1 C_2 coincide with E_4 E_1 E_2 the position of C_3 is given by

$$\mathbf{x} \bullet \mathbf{e}_4 = g_{34} \qquad \mathbf{x} \bullet \mathbf{e}_1 = g_{31} \qquad \mathbf{x} \bullet \mathbf{e}_2 = g_{32} \qquad \mathbf{x} \bullet \mathbf{x} = g_{33}$$

This locus consists of the two points in which a contour trisurface is cut by a straight line, one of the points being E_3 and the other its image in the triplane $\mathbf{e}_{124}$.

6. When we are setting up the **c**-lattice in its original position and have set up $\mathbf{c}_4$ $\mathbf{c}_1$ $\mathbf{c}_2$ there remains for $\mathbf{c}_3$ the choice between two positions which are images of one another in the triplane $\mathbf{c}_{124}$. In order that pure rotation may bring the **c**-lattice to coincidence with the **e**-lattice that position must be chosen for $\mathbf{c}_3$ that will bring it to coincidence with $\mathbf{e}_3$ and not with its image in $\mathbf{e}_{124}$.

7. As a special case let the **e** and **c** lattices be supercubical; let observer E have the prime vector $\mathbf{e}_4$ as world line while C has $\mathbf{c}_4$ as world line, so that each is at rest in his own system; let $\mathbf{c}_2$ coincide with $\mathbf{e}_2$, and $\mathbf{c}_3$ with $\mathbf{e}_3$. Let the classical velocity of C relative to E be v.

If the two lattices can be rotated to coincidence it is by rotation in the biplane $\mathbf{e}_{14}$.

We put

$$\mathbf{c}_4 = d^{\cdot 1}_{4}\mathbf{e}_1 + d^{\cdot 4}_{4}\mathbf{e}_4$$

and since on E's reckoning C does the distance $d^{\cdot 1}_{4}$ in the time $d^{\cdot 4}_{4}$ we have

$$d^{\cdot 1}_{4}/d^{\cdot 4}_{4} = v$$

whence

$$1 = \mathbf{c}_4 \cdot \mathbf{c}_4 = (d^{\cdot 4}_{4})^2(v^2 g_{11} + g_{44}) = (d^{\cdot 4}_{4})^2(1 - v^2)$$

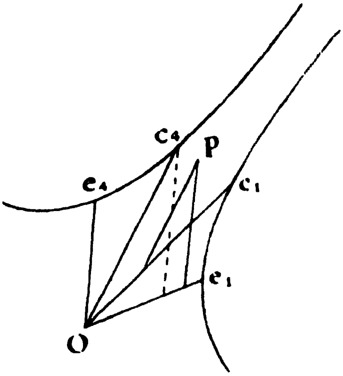


Figure for article 20,7.

and $\mathbf{c}_4 = d^{\cdot 4}_{4}(v\mathbf{e}_1 + \mathbf{e}_4)$

where $d^{\cdot 4}_{4} = (1 - v^2)^{-1/2}$

Since $\mathbf{c}_1$ is normal to $\mathbf{c}_4$ and in the biplane $\mathbf{e}_{14}$ its value is

$$\mathbf{c}_1 = d^{\cdot 4}_{4}(\mathbf{e}_1 + v\mathbf{e}_4)$$

If vector $\overrightarrow{OP}$ is $x^a\mathbf{e}_a$ in the **e**-lattice and $z^a\mathbf{c}_a$ in the **c**-lattice

$$x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + x^4\mathbf{e}_4 = z^1 d^{\cdot 4}_{4}(\mathbf{e}_1 + v\mathbf{e}_4) + z^2\mathbf{e}_2 + z^3\mathbf{e}_3 + z^4 d^{\cdot 4}_{4}(v\mathbf{e}_1 + \mathbf{e}_4)$$

so that

$$x^1 = d^{\cdot 4}_{4}(z^1 + vz^4)$$
$$x^2 = z^2$$
$$x^3 = z^3$$
$$x^4 = d^{\cdot 4}_{4}(z^4 + vz^1)$$

These equations are Lorentz's famous transformation. From *E*'s picture of the world which is expressed in terms of x^a these equations give the picture of the world as seen by *C* who has relative to *E* a velocity v along the x^1 axis and who expresses his picture in terms of z^a.

Lorentz first invented his transformation to give in terms of any known electromagnetic arrangement another possible electromagnetic arrangement. Later Minkowski saw that the two different arrangements could also be one single arrangement viewed by two different observers.

8. Let *E* and *C* be two observers, whose world lines are both straight. Let **k** be a vector along *E*'s world line and **l** one along *C*'s. How does *E* express in classical language his picture of *C*'s motion?

E splits the fourfold into time measured along **k** and space which is the triplane normal to **k**. Thus if from *L* the head of vector **l** we draw *LM* perpendicular to **k**, *E* says that the vector **l** is resolved into a time-lapse *OM* and a space vector $\overrightarrow{ML}$, or that *C* travels from *M* to *L* in time *OM*.

We know (article 5,7) that

$$\nu_{OM}\nu_{\mathbf{k}} = \varpi^2_{\mathbf{kl}} \qquad \nu_{ML}\nu_{\mathbf{k}} = \nu_{\mathbf{kl}}$$

and that $\varpi_{\mathbf{kl}}$ is positive and $\nu_{\mathbf{kl}}$ negative, so that in *E*'s picture *C* is travelling with a velocity of magnitude

$$\sqrt{-\nu_{\mathbf{kl}}}\,/\,\varpi_{\mathbf{kl}}$$

We note that the expression for *C*'s velocity is of degree zero in **k** and **l** and it is immaterial whether these vectors are the velocity vectors or other vectors with the same directions.

9. **Exercise.**—Show that in E's picture the classical velocity of C is given in direction and magnitude by the vector $\mathbf{l}\sigma_{\mathbf{k}}/\varpi_{\mathbf{kl}} - \mathbf{k}/\sigma_{\mathbf{k}}$ or ${}^1/_2\,\bar{k}\bar{l}^{\alpha\beta}\bar{k}\bar{\mathbf{e}}_{\alpha\beta}/\sigma_{\mathbf{k}}\varpi_{\mathbf{kl}}$, showing by direct calculation that the second expression has the appropriate size-number $\nu_{\mathbf{kl}}/\varpi_{\mathbf{kl}}^2$.

Is the Pointer of a Triplane Invariant?

10. In what circumstances is the pointer of a triplane invariant, that is to say when will our formula (article 11,11) used in a new lattice $\mathbf{c}_\alpha$ pick out the same normal to a triplane that is picked out by the formula used in the lattice $\mathbf{e}_\alpha$?

We first consider a particular case. As the prime vectors $\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$ $\mathbf{c}_4$ we take $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ $\mathbf{e}^4$. This requires the pointer of $\mathbf{e}^{123}$ when we are working in the $\mathbf{c}_\alpha$-lattice to lie along $\mathbf{e}_4$, but when we are working in the $\mathbf{e}_\alpha$-lattice it actually lies along $-\mathbf{e}_4$. Thus the pointer cannot be invariant for all transformations.

Again let the $\mathbf{e}$-lattice be supercubical, and for $\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$ $\mathbf{c}_4$ take $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $-\mathbf{e}_4$ which is congruent to the $\mathbf{e}$-lattice. The pointer of $\mathbf{e}_{123}$ in the $\mathbf{c}$-lattice is $\mathbf{c}_4$ or $-\mathbf{e}_4$ while in the $\mathbf{e}$-lattice it is $\mathbf{e}_4$. And the pointer cannot be invariant for all congruent transformations.

11. The pointer of triped $\mathbf{klm} \equiv {}^1/_6\,\bar{k}\bar{l}\bar{m}^{\alpha\beta\gamma}\mathbf{e}_{\alpha\beta\gamma}$ is $\sum\limits_{\alpha\beta\gamma\delta}{}^1/_3\,\overline{klm}^{\alpha\beta\gamma}\mathbf{e}^\delta\sqrt{-g}$ where the symbol $\sum\limits_{\alpha\beta\gamma\delta}$ directs summation for even permutations of α β γ δ. That will in future be the meaning of the symbol without further explanation. We shall consider transformation to a $\mathbf{c}$-lattice congruent to the $\mathbf{e}$-lattice. Let the two lattices be given in terms of one another by

$$\mathbf{e}_\alpha = f_\alpha^{\circ\theta}\mathbf{c}_\theta \qquad \mathbf{c}_\alpha = d_\alpha^{\circ\theta}\mathbf{e}_\theta$$

and let the vectors $k^a\mathbf{e}_a$ $l^a\mathbf{e}_a$ $m^a\mathbf{e}_a$ be $r^a\mathbf{c}_a$ $s^a\mathbf{c}_a$ $t^a\mathbf{c}_a$ when expressed in the new lattice.

From article 2 we see that the formulas we need for transformation from the **e** system to the **c** system are

$$\mathbf{e}_a = f_a^{\circ\theta}\mathbf{c}_\theta \quad \mathbf{e}^a = f_{\circ\theta}^a\mathbf{c}^\theta$$
$$k_a = a_{\circ a}^{\theta} r_\theta \quad k^a = d_\theta^{\circ a} r^\theta$$

12. For $\overline{klm}^{\beta\gamma\delta}$ we have

$$\overline{klm}^{\beta\gamma\delta} = \begin{vmatrix} d_\phi^{\circ\beta}r^\phi & d_\psi^{\circ\beta}s^\psi & d_\omega^{\circ\beta}t^\omega \\ d_\phi^{\circ\gamma}r^\phi & d_\psi^{\circ\gamma}s^\psi & d_\omega^{\circ\gamma}t^\omega \\ d_\phi^{\circ\delta}r^\phi & d_\psi^{\circ\delta}s^\psi & d_\omega^{\circ\delta}t^\omega \end{vmatrix}$$
$$= d_{\phi\psi\omega}{}^{\beta\gamma\delta} r^\phi s^\psi t^\omega$$

with the notation of article 16,3. For any set of values of ϕ ψ ω we take the six permutations and thus find

$$1/6\, d_{\phi\psi\omega}{}^{\beta\gamma\delta}\,\overline{rst}^{\phi\psi\omega}$$

In accordance with article 16,3 we replace $d_{\phi\psi\omega}{}^{\beta\gamma\delta}$ by $df_a^{\circ\theta}$ in which a β γ δ and θ ϕ ψ ω are even permutations and d is the determinant $|\ d_\phi^{\circ\sigma}\ |$. We then have

$$1/3\, d \sum_{\theta\phi\psi\omega} f_a^{\circ\theta}\,\overline{rst}^{\phi\psi\omega}$$

The numerical factor changes because the present summation is over 12 terms while the former sum had 24.

Hence for the pointer we have

$$1/3 \sum_{a\beta\gamma\delta} \overline{klm}^{\delta\gamma\beta}\mathbf{e}^a\sqrt{-g} = 1/3\, d \sum_{\theta\phi\psi\omega} \overline{rst}^{\omega\psi\phi} f_a^{\circ\theta} f_{\circ\rho}^{a}\,\mathbf{c}^\rho\sqrt{-g}$$

the numerical multiplier being the same on both sides because both sums are for 12 terms.

13. The operator **S** which converts $\mathbf{e}_a$ to $\mathbf{c}_a$, so that

$\mathbf{S}\cdot\mathbf{e}_\alpha = \mathbf{c}_\alpha$ is $\mathbf{S} = \mathbf{c}_\alpha\mathbf{e}^\alpha = f^{\alpha}_{\circ\theta}\mathbf{c}_\alpha\mathbf{c}^\theta$, and the conjugate operator is $\overline{\mathbf{S}} = \mathbf{e}_\alpha\mathbf{c}^\alpha = f^{\circ\phi}_{\alpha}\mathbf{c}_\phi\mathbf{c}^\alpha$. From these we have

$$\mathbf{S}\cdot\overline{\mathbf{S}} = f^{\alpha}_{\circ\theta}\mathbf{c}_\alpha(\mathbf{c}^\theta\cdot\mathbf{c}_\phi)f^{\circ\phi}_{\beta}\mathbf{c}^\beta = f^{\alpha}_{\circ\theta}f^{\circ\theta}_{\beta}\mathbf{c}_\alpha\mathbf{c}^\beta$$
$$\overline{\mathbf{S}}\cdot\mathbf{S} = f^{\circ\phi}_{\alpha}\mathbf{c}_\phi(\mathbf{c}^\alpha\cdot\mathbf{c}_\beta)f^{\beta}_{\circ\theta}\mathbf{c}^\theta = f^{\circ\phi}_{\alpha}f^{\alpha}_{\circ\theta}\mathbf{c}_\phi\mathbf{c}^\theta$$

And these are both the identity operator when $\mathbf{S}$ is a congruence operator, so that

$$f^{\alpha}_{\circ\theta}f^{\circ\theta}_{\beta} = f^{\circ\alpha}_{\theta}f^{\theta}_{\circ\beta} = \delta^{\alpha}_{\beta}$$

14. Thus $f^{\circ\theta}_{\alpha}f^{\alpha}_{\circ\rho}\mathbf{c}^\delta = \delta^\theta_\rho\mathbf{c}^\rho = \mathbf{c}^\theta$ and the relation between the pointers in the two lattices is

$$\tfrac{1}{3}\sum_{\alpha\beta\gamma\delta}\overline{klm}^{\alpha\beta\gamma}\mathbf{e}^\delta\sqrt{-g} = d\,\tfrac{1}{3}\sum_{\theta\phi\psi\omega}\overline{rst}^{\theta\phi\psi}\mathbf{c}^\omega\sqrt{-g}$$

The pointer is thus an invariant for transformations in which d is unity and not otherwise.

15. The prime tetrapeds of two congruent lattices have the same size-number, so that article 16,4 shows d^2 to be unity and d to be ± 1. If we require the transformation to be a rotation we can interpolate any number of congruent lattices, so that the change at any step is infinitesimal; in that case d must be unity for each step and therefore for the transformation.

Hence the pointer of a triped is invariant for all transformations that are rotations.

CHAPTER 21

A GRAVITATIONLESS REFERENCE SYSTEM

1. IT is a hypothesis of the relativity theory that a gravitational field acts on a ray of light as well as on a material particle. If we choose a man at rest on the earth as observer *E* his reference system would have a gravitational field and the path of a ray of light would be curved. Instead of this we choose as observer *E* a man falling freely under the earth's gravitation. His "local region" will extend vertically as far as the magnitude of the earth's gravitational force is sensibly equal to that at *E*'s origin. Horizontally it will include places for which the direction of the earth's attraction is sensibly parallel to that at the origin. This defines the spacelike extension of the fourfold local region; for its timelike extension we have time-lapses of corresponding magnitude.

2. In *E*'s local region as thus defined there is no gravitational field. Hence on our hypothesis the world line of a ray of light is straight.

3. In the local region straight lines, biplanes, and triplanes exist and *E* builds up his lattice of supercells with no warping of the fourfold to trouble him. Within the region the lattice quantities $g_{\alpha\beta}$ are constant, although beyond it they become variable.

4. We compare *E*'s picture of the world with that of

another observer *C*. The relation between *E* and *C* can be stated in two ways. In classical language we say that *E* and *C* are in relative motion with a constant relative velocity; in relativity language we may say that *C*'s reference system is obtained from *E*'s by a rotation in the fourfold about the origin.

5. This method of deriving *C*'s picture from *E*'s makes every coordinate in *C*'s lattice a homogeneous linear function of the coordinates of *E*'s lattice, so that a straight line for *E* is a straight line for *C*. Hence a ray of light, being straight for *E*, is also straight for *C*. Further the straightness of light rays being by hypothesis equivalent to the absence of a gravitational field, *C*'s lattice is also free from gravitation.

6. The question of the phenomena as seen by observer *E* beyond his local region belongs to the higher geometry of the fourfold. So does the question of the phenomena as seen by *C* when his system is derived from *E*'s otherwise than by uniform relative motion, when for example the relative velocity is accelerated. So does also the question of what happens when *E*'s system has a gravitational field. They are beyond the scope of this book.

7. So long as we confine ourselves to *E*'s system free from gravitation and to those of other observers *C* whose systems are derived by rotation in the fourfold, we know that the following quantities have the same measure in all systems or are invariant: the size-numbers of line vectors, of area vectors, of volume vectors, of supervolumes, all vector quantities and their outer products, the inner products of two line vectors, of two area vectors, of two volume vectors, and the pointer of a volume vector.

8. Apart from the value of the elementary geometry of the fourfold as a preparation for the higher geometry, this finding of invariants is the culmination of the whole discussion. For these invariants are the proper quantities in which to express such laws of mechanics and mathematical physics as can be stated for the local region, in order that these laws may be equally true for all lattice systems.

CHAPTER 22

MEASURE OF DISTANCE

1. AN event has a definite location in the fourfold. Its location may be given by four coordinates, or by a vector drawn to the event, or otherwise. A body or particle has not a location but has a one-dimensional locus, namely its world line. We can measure from one event to another, but we cannot measure from one locus to another, at least not until we have some agreement as to what that process is to mean.

2. The vector joining two events is called the separation between the events. If we think of the fourfold apart from any particular observer a separation is not a distance or a time-lapse or the sum of distance and time-lapse ; it is a new entity in which distance and time-lapse have lost their separate identity and are fused together.

3. For any particular observer the fourfold splits in two, into his time and his space. For him a separation usually consists of two elements, distance and time, that is to say one of the two events happens at some distance from the other and some time after it. In particular cases the time difference may be zero, that is to say the two events are simultaneous; the separation is then pure distance. And when we speak of distance in fourfold geometry we mean the separation between two simultaneous events.

4. When in the classical mode of thought we consider the distance between two bodies capable of relative motion we speak of "the distance at such and such an instant." That is to say that here also we are really measuring the distance between two events.

5. Consider two bodies S_1 S_2 at rest in P's system. Their world lines are parallel to P's velocity vector, which we will call $\overrightarrow{OU}$ and $\mathbf{u}$. Let their world lines and $\mathbf{u}$ lie in the same biplane, and in this biplane let the unit vector $\overrightarrow{OK} \equiv \mathbf{k}$ be drawn perpendicular to $\mathbf{u}$ and cutting the world lines of S_1 S_2 in X_1 X_2.

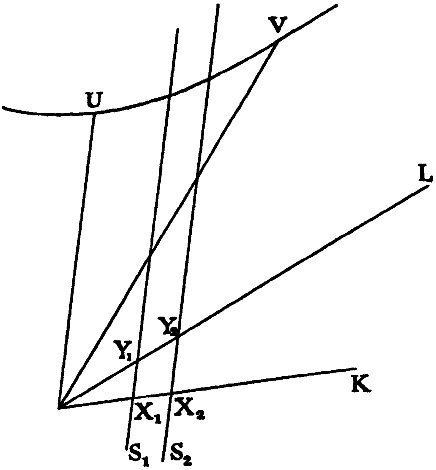


Figure for article 22,5.

For P time is zero all along OK, X_1 X_2 are simultaneous events, and $\sigma_{X_1X_2}$ is the distance between S_1 S_2 at time zero.

6. Now consider the situation as viewed by Q whose velocity vector $\overrightarrow{OV} \equiv \mathbf{v}$ also lies in the biplane $\mathbf{uk}$. In the

biplane we draw the unit vector $\overrightarrow{OL} \equiv \mathbf{l}$ perpendicular to OV. For Q time is zero all along **l**, and in particular at Y_1 Y_2 the events in which **l** intersects the world lines of S_1 S_2.

7. We shall see that the separations X_1X_2 Y_1Y_2 have different sizes, that is to say P and Q give different values to the distance between S_1 and S_2. This does not surprise if we reflect that each of them is measuring the separation between a pair of events, and that they select different pairs of events to measure.

8. Let $\overrightarrow{OU} \equiv \mathbf{u}$ $\overrightarrow{OV} \equiv \mathbf{v}$ be the velocity vectors of two observers P Q, and let **k** **l** be two unit vectors perpendicular to them in the biplane **uv**. Let UD VE UF VG as shown in the figure be parallel to **k** **l**. Let **k**

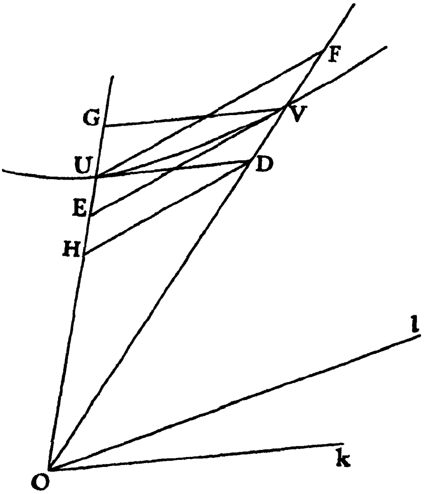


Figure for article 22,8.

head in the same direction as $\overrightarrow{UD}$, and $\mathbf{l}$ head so that if $\mathbf{u}$ $\mathbf{v}$ are brought to coincidence, so also are $\mathbf{k}$ $\mathbf{l}$.

9. We denote the inner product $\mathbf{u} \cdot \mathbf{v}$ by $\cosh \theta$. Then, the size-numbers of $\mathbf{u}$ $\mathbf{v}$ $\mathbf{k}$ $\mathbf{l}$ being $1 \quad 1 \quad -1 \quad -1$, we have

$$\overrightarrow{OG} = \mathbf{u} \cosh \theta \qquad \overrightarrow{OF} = \mathbf{v} \cosh \theta$$
$$\overrightarrow{OE} = \mathbf{u} \operatorname{sech} \theta \qquad \overrightarrow{OD} = \mathbf{v} \operatorname{sech} \theta$$

10. Also $\quad \nu_{GV} = \nu_{OV} - \nu_{OG} = 1 - \cosh^2 \theta$

so that, as $\overrightarrow{GV}$ and $\mathbf{k}$ head in the same direction, we take θ to be positive and have

$$\overrightarrow{GV} = \mathbf{k} \sinh \theta$$

whence

$$\overrightarrow{UD} = \mathbf{k} \tanh \theta$$

and similarly

$$\overrightarrow{UF} = \mathbf{l} \sinh \theta \qquad \overrightarrow{EV} = \mathbf{l} \tanh \theta$$

11. We have expressions for $\mathbf{v}$ in terms of $\mathbf{u}$ $\mathbf{k}$, and for $\mathbf{u}$ in terms of $\mathbf{v}$ $\mathbf{l}$, namely

$$\mathbf{v} = \overrightarrow{OG} + \overrightarrow{GV} = \mathbf{u} \cosh \theta + \mathbf{k} \sinh \theta$$
$$\mathbf{u} = \overrightarrow{OF} + \overrightarrow{FU} = \mathbf{v} \cosh \theta + \mathbf{l} \sinh \theta$$

12. The vector $\overrightarrow{OD}$, which is a piece of Q's world line, resolves itself, in P's opinion, into the time $\overrightarrow{OU}$ and the distance $\overrightarrow{UD}$. That is, in P's opinion, Q travels the distance $\tanh \theta$ in unit time and has classical velocity $\tanh \theta$.

13. Along vector $\mathbf{k}$ we lay off vectors $\overrightarrow{OX_1}$ $\overrightarrow{X_1X_2}$ $\overrightarrow{X_2X_3}$ . . each equal to $\overrightarrow{UD}$. Through X_1 X_2 X_3 . . . we draw the world lines of bodies T_1 T_2 T_3 . . . that are at rest relative

to P. Let P be an observer standing on a railway track. Then we may take $T_1\ T_2\ .\ .\ .$ to be telegraph poles set up along the track.

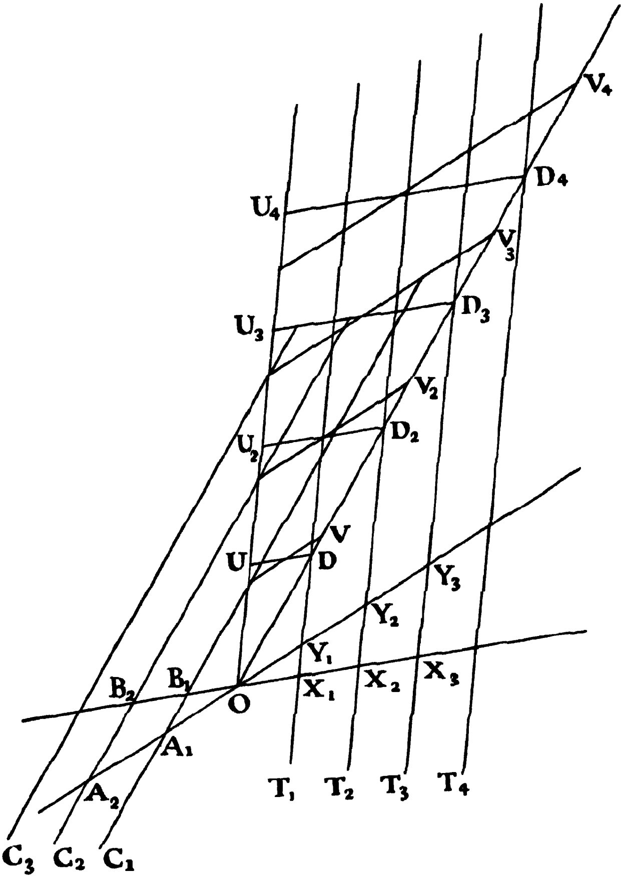


Figure for article 22,13.

Along P's world line we lay off $\overrightarrow{UU_2}\ \ \overrightarrow{U_2U_3}\ .\ .\ .$ equal to $\overrightarrow{OU}$ which is P's unit time. We draw $U_2D_2\ \ U_3D_3\ .\ .\ .$

parallel to **k** to meet the world lines of T_2 T_3 . . . in D_2 D_3. . . . Then the events D D_2 D_3 . . . are on Q's world line and are the coincidences of observer Q with the poles T_1 T_2 T_3. . . .

P's measure of the distance between two poles is the size of the separation between his simultaneous events X_1 and X_2; it is tanh θ. He says the time Q takes from one pole to the next is σ_{OU} or σ_{UU_2}, that is unit time, say one second.

14. Let the vector **l**, Q's line of zero time, meet the world lines of T_1 T_2 T_3 . . . in Y_1 Y_2 Y_3. . . . These events O Y_1 Y_2 . . . being simultaneous for Q, he takes the size of the separation of two of them as the distance between two poles. This is the size of DH in the figure of article 22,8 or tanh θ sech θ.

Q's time is measured along the vector **v**, OV being one second for him, so that his reckoning of the time-lapse from pole to pole is OD or sech θ.

Thus Q's reckoning of the distance between two poles and of the time-lapse between two poles is sech θ times P's reckoning.

15. Along vector **l** we lay off vectors $\overrightarrow{OA_1}$ $\overrightarrow{A_1A_2}$ . . . each equal to $\overrightarrow{VE}$ in the figure of article 22,8. Through A_1 A_2 . . . we draw the world lines of bodies C_1 C_2 . . . that are at rest relative to Q. We take Q to be sitting on the front of a train travelling along P's railway track, and we take C_1 C_2 . . . to be the couplings between successive coaches of the train.

The same kind of reasoning as before shows that for Q the length of a coach is VE or tanh θ and that a coach takes 1 second to pass P, while for P the length of a coach is $\tanh\theta\,\mathrm{sech}\,\theta$ and that it takes $\mathrm{sech}\,\theta$ seconds to pass him.

16. P's picture of the poles and coaches at his time 3, that is to say the intersection of U_3D_3 with the world lines, is shown in the upper of the figures. The lower figure shows Q's picture at what he calls time 3.

The reason why P and Q assign different values to distances and to time-lapses is now clear. While their phrases "distance between two places" and "time-lapse between two events" are identical, they mean quite different things to P and Q. To P the phrases mean the sizes of certain vectors, to Q they mean the sizes of quite other vectors.

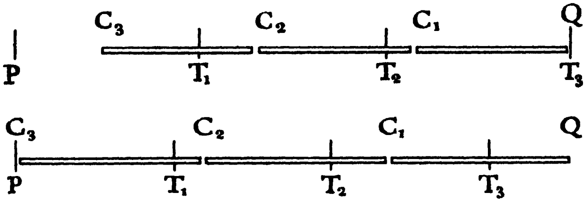


Figure for article 22,16.

For the purpose of the present illustration we may treat P's and Q's reference systems as gravitationless. For some purposes, for example to compare the time taken by a ray of light to make the circuit of a closed curve in the two directions, to ignore the gravitation would lead to error.

CHAPTER 23

COMPOSITION OF VELOCITIES

1. LET three observers P Q R have velocity vectors $\overrightarrow{OU} = \mathbf{u}$ $\overrightarrow{OV} = \mathbf{v}$ $\overrightarrow{OW} = \mathbf{w}$ which lie in the same biplane. In this biplane we take unit vectors **k** **l** **m** perpendicular to **u** **v** **w**. For each of the vectors **k** **l** **m** there is a choice of two

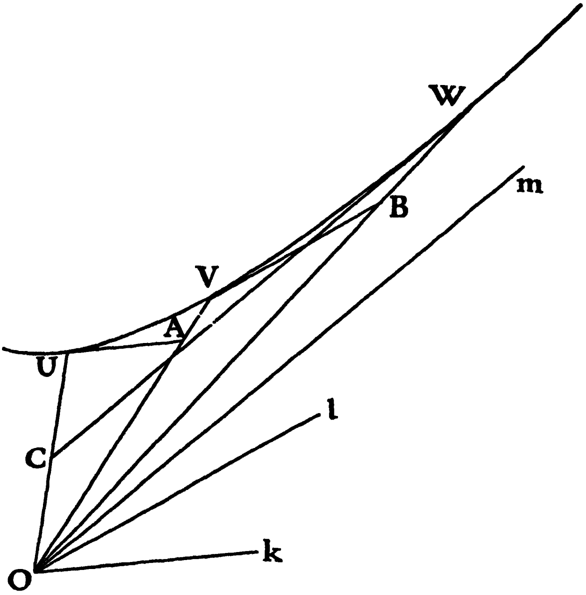


Figure for article 23,1.

opposite directions; we take them so that when $\mathbf{u}$ $\mathbf{v}$ $\mathbf{w}$ rotate to coincidence, so also do $\mathbf{k}$ $\mathbf{l}$ $\mathbf{m}$.

2. We draw UA VB WC parallel to $\mathbf{k}$ $\mathbf{l}$ $\mathbf{m}$ to meet $\mathbf{v}$ $\mathbf{w}$ $\mathbf{u}$ in A B C. We put

$$\mathbf{u} \cdot \mathbf{v} = \cosh\theta \qquad \mathbf{v} \cdot \mathbf{w} = \cosh\phi \qquad \mathbf{w} \cdot \mathbf{u} = \cosh\psi$$

Article 22,12 shows that the classical velocity of Q relative to P is σ_{UA} or $\tanh\theta$, that of R relative to Q σ_{VB} or $\tanh\phi$, and that of P relative to R σ_{WC} or $\tanh\psi$. The vectors $\overrightarrow{UA}$ $\overrightarrow{VB}$ $\overrightarrow{WC}$ we express as

$$\mathbf{k}\tanh\theta \qquad \mathbf{l}\tanh\phi \qquad \mathbf{m}\tanh\psi$$

so that for the figure we have drawn θ ϕ are positive and ψ negative.

3. From the symmetry of the figure in which $\mathbf{u}$ and $\mathbf{k}$, $\mathbf{v}$ and $\mathbf{l}$, $\mathbf{w}$ and $\mathbf{m}$, are three pairs of semidiameters of a hyperbola, the fact that (article 22,11)

$$\mathbf{v} = \mathbf{u}\cosh\theta + \mathbf{k}\sinh\theta$$

carries with it

$$\mathbf{l} = \mathbf{k}\cosh\theta + \mathbf{u}\sinh\theta$$

4. For $\mathbf{w}$ article 22,11 gives two expressions, namely

$$\mathbf{w} = \mathbf{u}\cosh\psi + \mathbf{k}\sinh(-\psi)$$
$$\mathbf{w} = \mathbf{v}\cosh\phi + \mathbf{l}\sinh\phi$$

In the first the minus sign enters with ψ because ψ is negative. In the second substitution of their values for $\mathbf{v}$ $\mathbf{l}$ gives

$$\begin{aligned}\mathbf{w} &= (\mathbf{u}\cosh\theta + \mathbf{k}\sinh\theta)\cosh\phi + (\mathbf{u}\sinh\theta + \mathbf{k}\cosh\theta)\sinh\phi \\ &= \mathbf{u}\cosh(\theta+\phi) + \mathbf{k}\sinh(\theta+\phi).\end{aligned}$$

Comparison of this with the first value for **w** gives

$$-\psi = \theta + \phi$$

or
$$\theta + \phi + \psi = 0$$

5. If we put $\alpha = \tanh\theta \quad \beta = \tanh\phi \quad \gamma = \tanh\psi$ the relation $\theta + \phi + \psi = 0$ gives

$$\alpha + \beta + \gamma + \alpha\beta\gamma = 0$$

that is to say; If, $P \quad Q \quad R$ remaining in a straight line, α is the classical velocity of Q relative to P, β that of R relative to Q, and γ that of P relative to R, these velocities are connected by the above relation.

6. Otherwise, to make all the quantities positive for our figure, we put $\alpha = \tanh\theta \quad \beta = \tanh\phi \quad \gamma = -\tanh\psi$ and have: If α is the velocity of Q relative to P and β that of R relative to Q, then the velocity of R relative to P is

$$\frac{\alpha + \beta}{1 + \alpha\beta}$$

7. We observe that if either of the quantities $\alpha \ \beta$ in the preceding article is unity, so also is $\frac{\alpha + \beta}{1 + \alpha\beta}$. That is to say if observer R moves with unit velocity (the velocity of light) relative to observer Q, he moves with the same velocity relative to observer P.

8. When the velocities are small $\alpha\beta$ vanishes compared with α and β, and we have the classical law for the addition of velocities, namely that if α is Q's velocity relative to P and β R's relative to Q, then R's velocity relat ve to P is $\alpha + \beta$.

9. We now come to the composition of two velocities not in the same line. Let $\overrightarrow{OU} \equiv \mathbf{u} \quad \overrightarrow{OV} \equiv \mathbf{v} \quad \overrightarrow{OW} \equiv \mathbf{w}$ be

three unit timelike vectors heading to the future. Let **k** be a unit vector in the biplane **uv** normal to **u**, and let **l** be a unit vector in the biplane **uw** normal to **u**. Let

$$\cosh\theta = \varpi_{\mathbf{uv}} \quad \cosh\phi = \varpi_{\mathbf{uw}} \quad \cosh\psi = \varpi_{\mathbf{vw}}$$

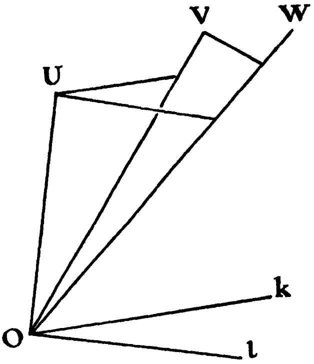


Figure for article 23,9.

10. Thus by article 22,11 we have

$$\mathbf{v} = \mathbf{u}\cosh\theta + \mathbf{k}\sinh\theta$$
$$\mathbf{w} = \mathbf{u}\cosh\phi + \mathbf{l}\sinh\phi$$

and these give

$$\cosh\psi = \mathbf{v}\cdot\mathbf{w} = \cosh\theta\cosh\phi + \varpi_{\mathbf{kl}}\sinh\theta\sinh\phi$$

The biplane **kl** is perpendicular to **u** and is spacelike. Hence (article 8,5) **k** and **l** meet at a Euclidian angle which we will call ω. Then $\mathbf{k}\cdot\mathbf{l} = -\cos\omega$ and our equation becomes

$$\cosh\psi = \cosh\theta\cosh\phi - \sinh\theta\sinh\phi\cos\omega$$

11. Let **u v w** be the velocity vectors of three observers *P Q R*. Then the classical velocities of *Q* and *R* relative

to P are $\tanh\theta$ and $\tanh\phi$, and ω is the angle between their directions; and $\tanh\psi$ is the relative velocity of Q and R.

We put $\tanh\theta = \alpha \quad \tanh\phi = \beta \quad \tanh\psi = \gamma$ and throw the equation into the form

$$\operatorname{sech}\psi = \frac{\operatorname{sech}\theta\operatorname{sech}\phi}{1 - \tanh\theta\tanh\phi\cos\omega}$$

or

$$1 - \gamma^2 = \frac{(1 - \alpha^2)(1 - \beta^2)}{(1 - \alpha\beta\cos\omega)^2}$$

and thus have γ, the relative velocity of Q and R, explicitly in terms of α and β, the velocities of Q and R relative to P, and the angle ω between them.

12. If we put $\beta = 1$ the formula of article 11 gives unity as the value of γ also, so that if R has unit velocity relative to Q, it has the same velocity relative to P. That is to say, if light travels with unit velocity relative to any gravitationless reference system, it also travels with unit velocity relative to a second system which has any constant velocity relative to the first.

13. **Exercise.**—Show that when the vectors **u v w** lie in the same biplane the formula of article 11 gives as the value of γ

$$\frac{\alpha + \beta}{1 + \alpha\beta} \quad \text{or} \quad \frac{\alpha - \beta}{1 - \alpha\beta} \quad \text{or} \quad \frac{\beta - \alpha}{1 - \alpha\beta}$$

14. **Exercise.**—Show that when $\alpha \; \beta \; \gamma$ are small the formula of article 11 becomes

$$\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos\omega$$

the formula of classical mechanics.

CHAPTER 24

THE GEODESIC

1. LET AB be a world line, that is a line whose direction at every point is timelike. Let P Q be neighbouring points on AB, the separation $\overrightarrow{PQ}$ being $d\mathbf{x}$. The size-number of $\overrightarrow{PQ}$ is

$$\nu_{PQ} = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$$

and the size of $\overrightarrow{PQ}$ or the interval of proper time from P to Q is

$$\sigma_{PQ} = \sqrt{g_{\alpha\beta}dx^{\alpha}dx^{\beta}}$$

The interval of proper time between A and B which we also call the size of the line is

$$\int_{A}^{B}\sqrt{g_{\alpha\beta}dx^{\alpha}dx^{\beta}}$$

Now while A and B are held fast let the world line be slightly varied. In general this variation will change the size of the line. When the line is such that no change takes place in the size it is called a geodesic.

2. Let P Q R be three points in succession along the curve, near together. Join PR and drop the perpendicular QM from Q on PR. M must lie between P and R as otherwise the world line would have a spacelike direction somewhere between P and R. Since QM is normal to PM

$$\nu_{PQ} = \nu_{PM} + \nu_{MQ}$$

Since MQ is spacelike ν_{MQ} is negative and equal to $-\sigma^2_{MQ}$ so that

$$\sigma^2_{PQ} = \sigma^2_{PM} - \sigma^2_{MQ}$$

and σ_{PQ} is less than σ_{PM}

In the same way σ_{QR} is less than σ_{MR}, so that the size of the line PQR is less than the size of PMR.

Thus by straightening the piece PQR of the line we have increased its size. And we can continue to increase the size by straightening until the whole line is straight, when the size is a maximum.

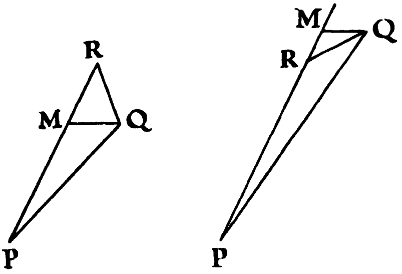


Figure for article 24,2.

We can state the result thus: Of all the possible world lines joining two points A B the interval of proper time is greatest when measured along the straight line.

3. Now let AB be a line whose direction is spacelike at every point, and let P Q be two neighbouring points on it.

In this case the size-number ν_{PQ} is negative and the size σ_{PQ} is

$$\sigma_{PQ} = \sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}$$

The size of the whole line AB is defined as the sum of the sizes of its elements, that is to say it is

$$\int_A^B \sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}$$

As before we define the geodesic joining A and B as any ine whose size is unaltered by infinitesimal variation the end points being held fast. Let P Q R be three points in succession on the line and near together, and let QM be the perpendicular dropped from Q on PR.

4. First we suppose the biplane PQR to be of mixed character, and QM consequently timelike. M falls between P and R as otherwise the line would have a piece of timelike direction. In this case.

$$\nu_{PQ} = -\sigma^2_{PQ} \qquad \nu_{PM} = -\sigma^2_{PM} \qquad \nu_{QM} = \sigma^2_{QM}$$

so that

$$-\sigma^2_{PQ} = -\sigma^2_{PM} + \sigma^2_{QM}$$

and σ_{PM} is greater than σ_{PQ}. We deal with QR MR in the same way and so find that the size of the straight piece PMR is greater than the size of the broken line PQR.

That is to say any straightening in a mixed biplane increases the size of the line AB.

Again suppose the biplane PQR spacelike, so that every line in it is spacelike, and let QM be perpendicular to PR as before. In this case it is immaterial whether M lies between P and R or not. All the size-numbers being negative we have

$$\sigma^2_{PQ} = \sigma^2_{PM} + \sigma^2_{MQ}$$
$$\sigma^2_{QR} = \sigma^2_{RM} + \sigma^2_{MQ}$$

from which, no matter how M lies,

$$\sigma_{PQ} + \sigma_{QR} > \sigma_{PR}$$

Thus any straightening in a spacelike biplane lessens the size of the line AB.

5. In the case of a spacelike line we find that the straight line has neither maximum nor minimum size, the size being increased by one kind of variation and diminished by another. But its size has a stationary value. Thus let PQ be a piece of a straight spacelike line, S a point in it, and TSU perpendicular to PQ. Then if the size of PTQ is greater than of PSQ, so also is the size of PUQ; if the former is less, so also is the latter. That is to say whatever effect a small distortion in one direction has, the reverse distortion has an effect of the same kind; and this is the condition for a stationary value.

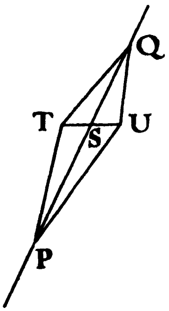


Figure for article 24,5.

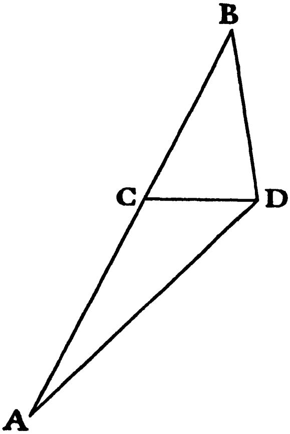


Figure for article 24,6.

6. Let the world line of an observer E be the straight line AB. From C the middle point of AB we draw CD at right angles, making its size v times the size of AC. Let the world line of an observer S consist of the two straight pieces AD and DB. This means that S travels away from E with speed v and then returns at the same speed.

Now

$$\sigma^2_{AD} = \sigma^2_{AC} - \sigma^2_{CD} = (1 - v^2)\sigma^2_{AC}$$
$$\sigma^2_{DB} = (1 - v^2,\sigma^2_{CB}$$

and S's interval of proper time from A to B is $\sigma_{AD} + \sigma_{DB}$ or $\sqrt{1 - v^2}\,\sigma_{AB}$.

Suppose the time that elapses for E from A to B to be an hour. Then the lapse of time for S is only $\sqrt{1 - v^2}$ hour. And if S travels with the speed of light, no time at all elapses for him.

7. **Exercise.**—We take a second as the unit of time. In order that the speed of light may be unity, 3×10^{10} centimetres must be equal to 1 second. On this basis show that the radius of the earth, $6{\cdot}371 \times 10^8$ cm., is $2{\cdot}124 \times 10^{-2}$ seconds, and that the velocity of a point on the equator relative to the earth's centre is $v = 1{\cdot}544 \times 10^{-6}$.

Let S be an observer on the equator and moving with the earth. Let E be an observer whose world line is straight (ignore the motion of the earth in its orbit), and let the world lines of S and E meet once a day. Show that the proper time of S between two meetings is $\sqrt{1 - v^2}$ times that of E, and that the difference between the two proper times is approximately 10^{-7} seconds.

8. This discussion of geodesics is for the local region, that is for a region for which the lattice functions $g_{\alpha\beta}$ are sensibly constant. In further regions the functions $g_{\alpha\beta}$ are not constant, and students who pursue the matter further will find that in such regions straight lines do not exist and that their place is taken by geodesics, defined as lines of stationary size.

There being no straight lines in the further region, the Euclidian definition of parallels falls to the ground, and the further development of the subject depends to a great extent on the discussion as to when two lines are to be called parallel. This discussion belongs to the third stage of relativity geometry.

CHAPTER 25

THE LAWS OF MOTION

1. THE restatement of the laws of motion is a question of mechanics and not of geometry. We shall nevertheless conclude by a discussion of the motion of a particle as an illustration of the virtue of the method of stating laws in terms of fourfold vectors.

We have seen that the velocity vector, which gives the change of position of a particle per unit of its proper time, is a quantity independent of the reference lattice used. Another such quantity is the vector that gives the change of the velocity vector per unit of proper time, and this is called the acceleration vector.

If the separation of the particle from the origin is $\mathbf{x}$ the velocity vector is $\mathbf{u} = \frac{d\mathbf{x}}{ds}$ in which s is the proper time, and the acceleration vector is

$$\mathbf{h} = \frac{d\mathbf{u}}{ds} = \frac{d^2\mathbf{x}}{ds^2}$$

2. We assume the mass of the particle to be invariant and denote it by m. Since $\mathbf{u}$ and m are invariant, the vector $m\mathbf{u}$ is also invariant; it is called the momentum vector. For the same reason the vector $m\mathbf{h}$ is invariant; this is the momentum acceleration vector.

We introduce another vector $\mathbf{f}$ called a force vector and put $m\mathbf{h} = \mathbf{f}$. The first three components of $\mathbf{f}$ correspond to the force of classical mechanics; what the fourth means remains to be seen.

3. In order to compare these four-dimensional quantities with the quantities of classical mechanics we take a super-cubical lattice with x^4 as time coordinate and x^1 x^2 x^3 as space coordinates. We put

$$\frac{dx^1}{dx^4} = v^1 \qquad \frac{dx^2}{dx^4} = v^2 \qquad \frac{dx^3}{dx^4} = v^3$$

$$v^1v^1 + v^2v^2 + v^3v^3 = vv$$

that is to say v^1 v^2 v^3 are the components of the velocity in classical mechanics, and its magnitude is v.

4. The expression for the components of the velocity vector in terms of the classical velocity are

$$u^1 = \frac{dx^1}{ds} = v^1/\sqrt{1 - vv} \qquad u^2 = v^2/\sqrt{1 - vv}$$

$$u^3 = v^3/\sqrt{1 - vv} \qquad u^4 = 1/\sqrt{1 - vv}$$

Consequently for the acceleration vector we have, for $a = 1 \ 2 \ 3$

$$h^a = \frac{du^a}{ds} = \frac{du^a}{dx^4}\Big/\frac{ds}{dx^4} = \frac{1}{\sqrt{1 - vv}}\,\frac{d}{dx^4}\left(\frac{v^1}{\sqrt{1 - vv}}\right)$$

and for the fourth component

$$h^4 = \frac{du^4}{dx^4}\Big/\frac{ds}{dx^4} = \frac{1}{\sqrt{1 - vv}}\,\frac{d}{dx^4}\left(\frac{1}{\sqrt{1 - vv}}\right)$$

5. Consequently the four components of the equation $m\mathbf{h} = \mathbf{f}$ are

$$\sqrt{1 - vv}\, f^a = \frac{d}{dx^4}\left(\frac{m}{\sqrt{1 - vv}}v^a\right) \qquad a = 1 \ 2 \ 3$$

$$\sqrt{1 - vv}\, f^4 = \frac{d}{dx^4}\left(\frac{m}{\sqrt{1 - vv}}\right)$$

When v is small the first three of these equations become the classical equations which say that the rate of change of momentum is equal to the applied force. Thus in these equations we have equations of invariant form which for the slow motions customary in mechanics reduce to the customary form. This is sufficient to justify the assumption of their universal validity as a hypothesis on which to work until it leads to some discrepancy.

6. If for all values of v these equations are to mean that the rate of change of momentum is equal to the applied force we must amend certain definitions. For the mass, defined as the quotient of the momentum by the velocity, we must use

$$\mu = m/\sqrt{1 - vv}$$

that is to say the mass is equal to m when the velocity is zero while for a finite velocity it depends upon v and is greater than m. Thus m is the mass in a reference lattice in which the particle is at rest; it is called the "proper mass" of the particle. For any other lattice the mass is $(1 - vv)^{-1/2}$ times the proper mass.

As the force components in classical mechanics we take the first three components of the fourfold force vector, each multiplied by $(1 - vv)^{1/2}$. The equations have then the desired meaning, namely that the rate of change of momentum is equal to the applied force.

In confirmation of this somewhat startling departure from the classical laws of motion we have the experimental fact that for rapidly moving electrons the increase in momentum is greater than in proportion to the increase of velocity.

7. We have still to consider the fourth equation of article 5.

That the size-number of velocity **u** is unity is expressed by the equation

$$u^4u^4 - u^1u^1 - u^2u^2 - u^3u^3 = 1$$

Differentiation with respect to the proper time s gives

$$u^4h^4 - u^1h^1 - u^2h^2 - u^3h^3 = 0$$

or since $m\mathbf{h} = \mathbf{f}$

$$u^4f^4 = u^1f^1 + u^2f^2 + u^3f^3$$

or

$$\sqrt{1 - vv}f^4 = v^1(\sqrt{1 - vv}f^1) + v^2(\sqrt{1 - vv}f^2) + v^3(\sqrt{1 - vv}f^3)$$

The right side expresses the rate at which work is being done on the particle by the applied force, or the rate of increase of kinetic energy. And since the left side is by article 5 equal to $\frac{d}{dx^4}\{m(1 - vv)^{-1/2}\}$ the quantity $m(1 - vv)^{-1/2}$ only differs from the kinetic energy by an additive constant.

To the second power of v we have

$$m\{(1 - vv)^{-1/2} - 1\} = {}^1/_2\, mv^2$$

giving the classical expression for the kinetic energy. Thus in

$$m\{(1 - vv)^{-1/2} - 1\}$$

we have an amended expression for the kinetic energy, and the fourth equation of article 5 expresses the fact that the work done by the applied force is equal to the increase of kinetic energy.

APPENDIX

NOTE ON THE TETRAPED

FROM four vectors **k l m n** we can make 24 tetrapeds by different arrangements of the vectors (or we could say that there are 24 ways of naming the one tetraped). We take one tetraped **klmn** and name it f. Since (article 1,19) interchange of two vectors changes the sign of the tetraped, there are 12 tetrapeds equal to f, namely the 12 obtained by even permutations from **klmn**. The other 12 obtained by odd permutations are equal to $-f$.

The size σ_f expresses magnitude only without regard to algebraic sign, all 24 tetrapeds have the same size, and f is either equal to σ_f or to $-\sigma_f$. The size-number ν_f is always equal to $-\sigma_f^2$, and therefore also to $-f^2$.

Of the two prime tetrapeds $\mathbf{e}_{1234}$ $\mathbf{e}^{1234}$ one must be positive and the other negative since $\mathbf{e}_{1234}$ is equal to $g\mathbf{e}^{1234}$ and g is negative. It is a matter of convention which we make positive, and we shall choose $\mathbf{e}_{1234}$. Then since the size of $\mathbf{e}_{1234}$ is $\sqrt{-g}$ and that of $\mathbf{e}^{1234}$ is $1/\sqrt{-g}$ we have

$$\mathbf{e}_{1234} = \sqrt{-g} \qquad \mathbf{e}^{1234} = -1/\sqrt{-g}$$

Using f^{1234} to denote $\overline{klmn}^{1234}$ we have

$$f = \mathbf{klmn} = \overline{klmn}^{1234}\mathbf{e}_{1234} = f^{1234}\sqrt{-g}$$

and also $$f = \overline{klmn}_{1234}\mathbf{e}^{1234} = -f_{1234}/\sqrt{-g}$$

These are the results of article 12,9 in which $\mathbf{e}_{1234}$ has been assumed to be positive.

ANSWERS TO EXERCISES

Each exercise is denoted by its chapter and article numbers

2,6. The vectors $\pm \mathbf{e}_1 \quad \pm \mathbf{e}_2 \quad \pm \mathbf{e}_3 \quad \pm \mathbf{e}_4$ furnish four edges for each supercell that has a corner at O; the choice of signs gives 2^4 or 16 supercells. The number of supercells with $\mathbf{e}_1$ as an edge is 2^3 or 8; the number with $\mathbf{e}_{12}$ as a biface is 4; and the number with $\mathbf{e}_{123}$ as a triface is 2.

2,7. The coordinates of the sixteen centres are

$$\pm {}^1/_2 \quad \pm {}^1/_2 \quad \pm {}^1/_2 \quad \pm {}^1/_2$$

The triplane $x^1 = {}^1/_2$ contains eight of them and $x^1 = - {}^1/_2$ contains the other eight. With this pair of triplanes parallel to $\mathbf{e}_{234}$ we take three other pairs parallel to the other three prime volume vectors. The eight are the boundaries of a tetraped enclosing the origin.

The only corner inside the enclosure is the origin.

For edges, the supercell $\mathbf{e}_{1234}$ (like every other supercell of the sixteen) has the halves of four edges inside, and each of them is shared by 8 supercells, so these half edges make up ${}^1/_2 \times 4 \times 16 \div 8$ or 4 whole edges.

Each supercell of the lattice has a quarter of each of 6 bifaces inside, each biface shared by 4 supercells. So the number of whole bifaces is ${}^1/_4 \times 6 \times 16 \div 4$ or 6.

Each supercell has an eighth of each of 4 trifaces inside, each triface shared by 2 supercells. In all ${}^1/_8 \times 4 \times 16 \div 2$ or 4 whole trifaces.

2,19. The six projections of P are $V \quad A \quad B \quad C \quad D \quad E$, the letters that are not shown in the figure being so placed that $OUAX$ $OUBY$ $OWCX$ $OXDY$ $OYEW$ are parallelograms. For these six points lie in the appropriate biplanes, while PT TV are parallel to $\mathbf{e}_{34}$, PT TA parallel to $\mathbf{e}_{24}$, PS SB to $\mathbf{e}_{23}$, PT TC to $\mathbf{e}_{14}$, PQ QD to $\mathbf{e}_{12}$, and PQ QE to $\mathbf{e}_{13}$.

The projection on $\mathbf{e}_{12}$ of $\mathbf{x}$ is $x^1\mathbf{e}_1 + x^2\mathbf{e}_2$, for it is $x^3\mathbf{e}_3 + x^4\mathbf{e}_4$ a vector parallel to $\mathbf{e}_{34}$ that we must compound with it to give $\mathbf{x}$. And so for the other projections.

4,13. The only limitation on the prime vectors is that they must be independent. We chose $\mathbf{e}_1$. Then $\mathbf{e}_2$ must not coincide with $\mathbf{e}_1$. Then $\mathbf{e}_3$ must not lie in the biplane $\mathbf{e}_{12}$. And $\mathbf{e}_4$ must not lie in the triplane $\mathbf{e}_{123}$. Each of the four vectors can thus be made timelike or spacelike as we please.

5,3. $\mathbf{e}^1 \cdot \mathbf{e}_1 = (g^{11}\mathbf{e}_1 + g^{12}\mathbf{e}_2)\cdot\mathbf{e}_1 = g^{11}\mathbf{e}_1\cdot\mathbf{e}_1 + g^{12}\mathbf{e}_2\cdot\mathbf{e}_1$
$= g^{11}g_{11} + g^{12}g_{12}$

and so for $\mathbf{e}^1 \cdot \mathbf{e}_2$.

5,8. The condition is that it should be possible to choose λ μ such so as to make $\nu_{\lambda\mathbf{l}+\mu\mathbf{m}}$ positive. This expression is

$$\lambda^2\nu_{\mathbf{l}} + 2\lambda\mu\varpi_{\mathbf{lm}} + \mu^2\nu_{\mathbf{m}}$$

or

$$\{(\lambda\nu_{\mathbf{l}} + \mu\varpi_{\mathbf{lm}})^2 + \mu^2(\nu_{\mathbf{l}}\nu_{\mathbf{m}} - \varpi_{\mathbf{lm}}^2)\}/\nu_{\mathbf{l}}$$

The numerator is capable of negative values, and the expression capable of positive values, when $\varpi_{\mathbf{lm}}^2$ is greater than $\nu_{\mathbf{l}}\nu_{\mathbf{m}}$.

5,9. If the parallelogram **lm** is spacelike both the base **l** and the height **h** are spacelike, so that $\nu_{\mathbf{l}}$ $\nu_{\mathbf{h}}$ are both negative, and $\nu_{\mathbf{lm}} \equiv \nu_{\mathbf{l}}\nu_{\mathbf{h}}$ is positive. If the parallelogram is mixed, of base and height one must be timelike and one spacelike, so that $\nu_{\mathbf{lm}}$ must be negative.

6,11. The prime vectors of the derived lattice of $\mathbf{e}_{12}$ lie in the biplane $\mathbf{e}_{12}$ and will thus coincide with $\mathbf{e}^1$ $\mathbf{e}^2$ only if $\mathbf{e}_3$ is perpendicular to $\mathbf{e}_{12}$.

7,2. The prime vectors of the derived lattice of $\mathbf{e}_{123}$ lie in the triplane $\mathbf{e}_{123}$. $\mathbf{e}^1$ $\mathbf{e}^2$ $\mathbf{e}^3$ are perpendicular to $\mathbf{e}_4$ and will thus lie in the triplane $\mathbf{e}_{123}$ only if $\mathbf{e}_4$ is perpendicular to $\mathbf{e}_{123}$.

9,8. Let O be the origin and E_α the head of vector $\mathbf{e}_\alpha$. Let U be the projection of E_4 on the triplane $\mathbf{e}_{123}$, so that the height of the five-face is $\mathbf{h} \equiv \overrightarrow{UE_4}$. Let a triplane parallel

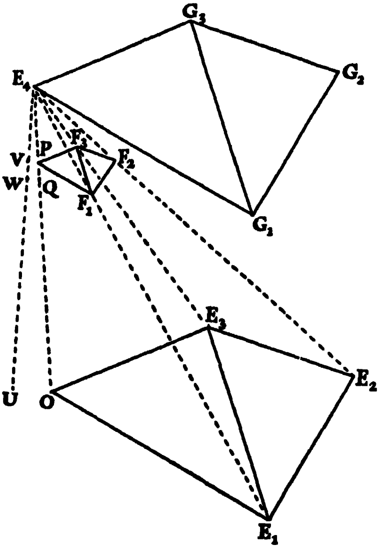


Figure for answer to exercise 9,8.

to the base $B \equiv OE_1E_2E_3$ travel from E_4 to O. When it cuts $\mathbf{e}_4$ at P and $\mathbf{h}$ at V let $\overrightarrow{PE_4} = \lambda\mathbf{e}_4$ so that also $\overrightarrow{VE_4} = \lambda\mathbf{h}$; let Q W correspond for the value $\lambda + d\lambda$ to P V.

If the triplane through P cuts the five-face in the section $S \equiv PF_1F_2F_3$ we shall have

$$\overrightarrow{PF_1} = \lambda\mathbf{e}_1 \qquad \overrightarrow{PF_2} = \lambda\mathbf{e}_2 \qquad \overrightarrow{PF_3} = \lambda\mathbf{e}_3$$

and S is a copy of the base B on the scale $\lambda : 1$.

The size of the supervolume generated by shifting the triped $\mathbf{e}_{123}$ along $\mathbf{e}_4$ is $\sigma_{\mathbf{e}_{1234}}$ or $\sigma_{\mathbf{e}_{123}}\sigma_{\mathbf{h}}$. Since $\sigma_B = {}^1/_6\,\sigma_{\mathbf{e}_{123}}$ the size of the superprism generated by the shift of B along $\mathbf{e}_4$ is $\sigma_B\sigma_{\mathbf{h}}$ or ${}^1/_6\,\sigma_{\mathbf{e}_{1234}}$.

The element of supervolume generated by section S in moving from position λ to position $\lambda + d\lambda$, or the element of the five-face traversed by the triplane from λ to $\lambda + d\lambda$, has size $\sigma_S\sigma_{VW}$ or $\lambda^3\sigma_B d\lambda\sigma_{\mathbf{h}}$. To obtain the size of the five-face we replace $\lambda^3 d\lambda$ by its integral $\int_0^1 \lambda^3 d\lambda$ or ${}^1/_4$. Thus the size of the five-face is one quarter of $\sigma_B\sigma_{\mathbf{h}}$ that is a quarter of the superprism generated by shifting the tetrahedron with edges $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ along $\mathbf{e}_4$.

9,9. With the notation of the preceding exercise the five-face is ${}^1/_4\,\sigma_B\sigma_{\mathbf{h}}$ and $\sigma_B\sigma_{\mathbf{h}}$ is ${}^1/_6\,\sigma_{\mathbf{e}_{1234}}$, whence the required result.

9,10. For the superprism we take that generated (in the figure for the answer to 9,8) by the shift of $B \equiv OE_1E_2E_3$ along $\mathbf{e}_4$ to $E_4G_1G_2G_3$. We consider the four five-faces

$$\begin{aligned} K &\equiv E_4OE_1E_2E_3 \\ L &\equiv E_4G_1E_1E_2E_3 \\ M &\equiv E_4G_1G_2E_2E_3 \\ N &\equiv E_4G_1G_2G_3E_2 \end{aligned}$$

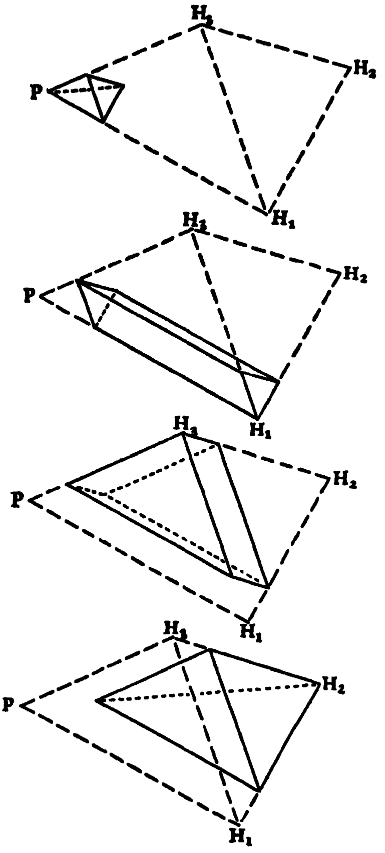
each being named by its five corners. We consider the volumes of intersection of these five-faces by the triplane T whose equation is $x^4 = \kappa$. T meets four of the ten edges of the five-face K in points whose coordinates are

$$0\ 0\ 0\ \kappa \qquad \lambda\ 0\ 0\ \kappa \qquad 0\ \lambda\ 0\ \kappa \qquad 0\ 0\ \lambda\ \kappa$$

being used to mean $1 - \kappa$. These are the corners of the figure of intersection of T and K, which is a tetrahedron.

T meets six of the edges of five-face L and the coordinates of the points of intersection are

$$\begin{array}{ccc} \lambda\,0\,0\,\kappa & 0\,\lambda\,0\,\kappa & 0\,0\,\lambda\,\kappa \\ 1\,0\,0\,\kappa & \kappa\,\lambda\,0\,\kappa & \kappa\,0\,\lambda\,\kappa \end{array}$$



Figures for answer to exercise 9,10.

These are the corners of a triangular prism which is the intersection of T and L. Similarly T intersects M in a triangular prism whose corners are at

$$\begin{matrix} 0\,0\,1\,\kappa & 0\,0\,\lambda\,\kappa & \kappa\,0\,\lambda\,\kappa \\ 0\,\lambda\,\kappa\,\kappa & 0\,\lambda\,0\,\kappa & \kappa\,\lambda\,0\,\kappa \end{matrix}$$

and intersects N in a tetrahedron whose corners are at

$$0\,1\,0\,\kappa \qquad 0\,\lambda\,0\,\kappa \qquad \kappa\,\lambda\,0\,\kappa \qquad 0\,\lambda\,\kappa\,\kappa$$

The figures show the four intersections, the tetrahedron $PH_1H_2H_3$ being the intersection of T with the superprism. Thus the intersections of T with the five-faces build up its intersection with the superprism. This being true for all values of κ we see that the five-faces K L M N build up the superprism.

The section of K by T, being one sixth of the triped of edges $\lambda\mathbf{e}_1$ $\lambda\mathbf{e}_2$ $\lambda\mathbf{e}_3$, is

$$^1/_6\,(1-\kappa)^3\mathbf{e}_{123}$$

in which $\mathbf{e}_{123}$ means the triped with edges $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$. This section by shifting through $d\kappa\mathbf{e}_4$ generates the element of supervolume

$$^1/_6(1-\kappa)^3 d\kappa\mathbf{e}_{1234} \qquad \text{or} \qquad ^1/_6(1-\kappa)^3 d\kappa\sqrt{-g}$$

We thus have for the five-face κ

$$^1/_6\sqrt{-g}\int_0^1(1-\kappa)^3 d\kappa \qquad \text{or} \qquad \sqrt{-g}/24$$

The prism in which T cuts the five-face L has edges

$$\kappa\mathbf{e}_1 \quad \lambda(\mathbf{e}_2-\mathbf{e}_1) \quad \lambda(\mathbf{e}_3-\mathbf{e}_1)$$

and by shifting its corners along the parallel edges we obtain an equal prism with edges

$$\kappa\mathbf{e}_1 \quad \lambda\mathbf{e}_2 \quad \lambda\mathbf{e}_3$$

The vector expression for the section is thus $^1/_2\kappa(1-\kappa)^2\mathbf{e}_{123}$. A shift along $d\kappa\mathbf{e}_4$ generates the supervolume $^1/_2\kappa(1-\kappa)^2 d\kappa\mathbf{e}_{1234}$, and the supervolume L is

$$^1/_2\sqrt{-g}\int_0^1\kappa(1-\kappa)^2 d\kappa = \sqrt{-g}/24$$

The section of M has edges $\kappa\mathbf{e}_1$ $\lambda(\mathbf{e}_2 - \mathbf{e}_3)$ $\kappa\mathbf{e}_3$, its vector expression is ${}^1/_2\kappa^2(1 - \kappa)\mathbf{e}_{123}$ and

$$M = {}^1/_2\,\mathbf{e}_{1234}\int_0^1 \kappa^2(1 - \kappa)d\kappa = \sqrt{-g}/24$$

The section of N has edges $\kappa\mathbf{e}_1$ $\kappa\mathbf{e}_2$ $\kappa\mathbf{e}_3$ and vector expression ${}^1/_6\,\kappa^3\mathbf{e}_{123}$ so that

$$N = {}^1/_6\,\mathbf{e}_{1234}\int_0^1 \kappa^3 d\kappa = \sqrt{-g}/24$$

Thus the four five-faces into which we cut up the super-prism are equal.

The equality of the five-faces can also be shown geometrically. Two five-faces on the same tetrahedral base and of the same height are equal because the two tetrahedra in which they are cut by any triplane parallel to the base are congruent. The five-face $E_4G_1E_1E_2E_3$ is equal to $OG_1E_1E_2E_3$ because E_4O the join of the vertices is parallel to G_1E_1 which lies in the common base $G_1E_1E_2E_3$; and $OG_1E_1E_2E_3$ is equal to $OE_4E_1E_2E_3$ because G_1E_4 the join of their vertices is parallel to E_1O which lies in their common base. Similarly

$$E_4G_1G_2E_2E_3 = OG_1G_2E_2E_3 = OE_1G_2E_2E_3 = OE_1E_4E_2E_3$$

and

$$\begin{aligned} E_4G_1G_2G_3E_3 = OG_1G_2G_3E_3 &= OE_1G_2G_3E_3 \\ &= OE_1E_2G_3E_3 = OE_1E_2E_4E_3 \end{aligned}$$

That is to say the first five-face is equal to each of the other three.

11,3. If the corners of the circuit are P Q R S, the film may consist of triangles OPQ OQR ORS.

11,9. First we put $\mathbf{e}^3 = g^{3\alpha}\mathbf{e}_\alpha$ $\mathbf{e}^4 = g^{4\alpha}\mathbf{e}_\alpha$ and obtain

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}^3\mathbf{e}^4 = \mathbf{e}_1\mathbf{e}_2(g^{33}g^{44}\mathbf{e}_3\mathbf{e}_4 + g^{34}g^{34}\mathbf{e}_4\mathbf{e}_3) = g^{3434}\mathbf{e}_{1234} = \sqrt{-g}\,g^{3434}.$$

Using the derived lattice we obtain $g_{1212}\mathbf{e}^{1234} = -g_{1212}/\sqrt{-g}$ So that

$$g_{1212} = gg^{3434}$$

11,10. $\mathbf{e}_1\mathbf{e}_2\mathbf{e}^\alpha\mathbf{e}^\beta = \mathbf{e}_1\mathbf{e}_2 g^{\alpha\theta}g^{\beta\phi}\mathbf{e}_\theta\mathbf{e}_\phi$ which expression gives finite terms only when $\theta = 3 \quad \phi = 4$ and when $\theta = 4 \quad \phi = 3$, so that its value is $g^{\alpha\beta 34}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$. Also the given expression is equal to $g_{1\theta}g_{2\phi}\mathbf{e}^\theta\mathbf{e}^\phi\mathbf{e}^\alpha\mathbf{e}^\beta$, or, since $\theta \quad \phi \quad \alpha \quad \beta$ must be unequal to give finite terms, equal to $g_{12\gamma\delta}\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3\mathbf{e}^4$ in which $\gamma \quad \delta$ are such as to make $\alpha \quad \beta \quad \gamma \quad \delta$ an even permutation. This gives the result stated.

12,5. Let us assume $l^4 = 0 \quad m^1 = 0$. Then of the six equations $\overline{lm}^{\rho\sigma} = a^{\rho\sigma}$ five take the form

$l^1m^2 = a^{12} \quad l^1m^3 = a^{13} \quad l^1m^4 = a^{14} \quad l^2m^4 = a^{24} \quad l^3m^4 = a^{34}$

which can be looked on as giving $m^2 \quad m^3 \quad m^4 \quad l^2 \quad l^3$ in terms of l^1. With these values substituted the sixth equation, $\overline{lm}^{23} = a^{23}$ becomes $a^{12}a^{34} + a^{23}a^{14} + a^{31}a^{24} = 0$. Thus the vectors $\mathbf{l} \quad \mathbf{m}$ which we have determined satisfy all the conditions.

12,11. The determinant $\overline{lmnp}^{1234}$ can be expanded as

$$\overline{lm}^{23}\overline{np}^{14} + \overline{lm}^{31}\overline{np}^{24} + \overline{lm}^{12}\overline{np}^{34}$$
$$+ \overline{lm}^{14}\overline{np}^{23} + \overline{lm}^{24}\overline{np}^{31} + \overline{lm}^{34}\overline{np}^{12}$$

whence the result.

12,12. $\mathbf{e}^\alpha\mathbf{e}^\beta\mathbf{e}_\rho\mathbf{e}_\sigma$ is equal to $g^{\alpha\theta}g^{\beta\phi}\mathbf{e}_\theta\mathbf{e}_\phi\mathbf{e}_\rho\mathbf{e}_\sigma$ and by interchanging $\theta \quad \phi$ to $-g^{\alpha\phi}g^{\beta\theta}\mathbf{e}_{\theta\phi\rho\sigma}$ and so by addition to

$$^1/_2\, g^{\alpha\beta\theta\phi}\mathbf{e}_{\theta\phi\rho\sigma}$$

We take $\rho \quad \sigma \quad \tau \quad \omega$ to be an even permutation from $1 \quad 2 \quad 3 \quad 4$. Our expression contains two finite terms which are given by $\theta = \tau \quad \phi = \omega$ and $\theta = \omega \quad \phi = \tau$, so that the expression is

$$g^{\alpha\beta\tau\omega}\mathbf{e}_{\tau\omega\rho\sigma} \qquad \text{or} \qquad g^{\alpha\beta\tau\omega}\sqrt{-g}$$

$\mathbf{e}_{1234}$ being assumed to be positive.
Similarly, $\mathbf{e}^{1234}$ being negative, we obtain

$$-g_{\rho\sigma\gamma\delta}/\sqrt{-g}$$

where $\alpha \quad \beta \quad \gamma \quad \delta$ are an even permutation, and the equating of the two values gives the required result.

12,13. The determinant

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{\rho 1} & g_{\rho 2} & g_{\rho 3} & g_{\rho 4} \\ g_{\sigma 1} & g_{\sigma 2} & g_{\sigma 3} & g_{\sigma 4} \end{vmatrix}$$

is equal to the given expression, and has the required values for the various values of $\rho \quad \sigma$.

Again $\overline{rs}^{12} = \begin{vmatrix} r^1 & r^2 \\ s^1 & s^2 \end{vmatrix} = \begin{vmatrix} g^{1\kappa} & g^{2\kappa} \\ g^{1\lambda} & g^{2\lambda} \end{vmatrix} r_\kappa s_\lambda = {}^1\!/_2 g^{12\kappa\lambda}\overline{rs}_{\kappa\lambda}$

$$= {}^1\!/_2 g^{12\alpha\beta}\sqrt{-g}\,\overline{uv}^{\gamma\delta}$$

and $\qquad \overline{uv}^{\gamma\delta} = {}^1\!/_2\, g^{\gamma\delta\kappa\lambda}\overline{uv}_{\kappa\lambda}$

so that $\quad \overline{rs}^{12} = {}^1\!/_4\sqrt{-g}g^{12\alpha\beta}g^{\gamma\delta\kappa\lambda}\overline{uv}_{\kappa\lambda}$

The first part of the exercise shows that the finite terms of this expression are given by

$$\kappa = 3 \quad \lambda = 4 \quad \text{and} \quad \kappa = 4 \quad \lambda = 3,$$

so that we have

$$\overline{rs}^{12} = {}^1\!/_2\sqrt{-g}\{\overline{uv}_{34}/g - \overline{uv}_{43}/g\}$$
$$= -\overline{uv}_{34}/\sqrt{-g}$$

which is the result stated.

13,5. The two processes give the same result, namely

$$\mathbf{k}\varpi_{\mathbf{lr}}\varpi_{\mathbf{sx}} + \mathbf{k}\varpi_{\mathbf{lt}}\varpi_{\mathbf{ux}} + \mathbf{m}\varpi_{\mathbf{nr}}\varpi_{\mathbf{sx}} + \mathbf{m}\varpi_{\mathbf{nt}}\varpi_{\mathbf{ux}} + \ \cdot\ \cdot\ \cdot$$

13,8. $a^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta \cdot \mathbf{x} = a^{\alpha\beta}\mathbf{e}_\alpha x_\beta$

$$= \Sigma k^\alpha l^\beta \mathbf{e}_\alpha x_\beta = \Sigma k^\alpha \mathbf{e}_\alpha l^\beta x_\beta = \Sigma \mathbf{k}\varpi_{\mathbf{lx}}$$

which is the expression of article 2.

13,17. We determine $\mathbf{l} \quad \mathbf{m}$ as in exercise 12,5 so that $\mathbf{lm}$ is equal to the area vector ${}^1\!/_2 a^{\alpha\beta}\mathbf{e}_{\alpha\beta}$. Then $a^{\alpha\beta}$ is $\overline{lm}^{\alpha\beta}$ and

$$\mathbf{A} = \overline{lm}^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta = l^\alpha\mathbf{e}_\alpha m^\beta\mathbf{e}_\beta - m^\alpha\mathbf{e}_\alpha l^\beta\mathbf{e}_\beta = \mathbf{lm} - \mathbf{ml}$$

14,9. The vanishing of $\mathbf{a}.\mathbf{b}$ is the condition that each area vector contains a direction that is perpendicular to the other.

We take $\mathbf{a}$ and $\mathbf{b}$ in the forms $\mathbf{kl}$ and $\mathbf{mn}$. We put $\mathbf{r} = \kappa\mathbf{k} + \lambda\mathbf{l}$ $\mathbf{s} = \mu\mathbf{m} + \nu\mathbf{n}$ so that $\mathbf{r}$ lies in $\mathbf{a}$ and $\mathbf{s}$ in $\mathbf{b}$. $\mathbf{s}$ is perpendicular to $\mathbf{a}$ if $\mathbf{r}.\mathbf{s}$ vanishes for all values of of κ λ, that is if

$$\mathbf{k}.(\mu\mathbf{m} + \nu\mathbf{n}) = 0 \qquad \mathbf{l}.(\mu\mathbf{m} + \nu\mathbf{n}) = 0$$

and there is a vector $\mathbf{s}$ that lies in $\mathbf{b}$ and is perpendicular to $\mathbf{a}$ if it is possible to choose μ ν so as to satisfy these equations. The condition for this is

$$0 = \begin{vmatrix} \mathbf{k}.\mathbf{m} & \mathbf{k}.\mathbf{n} \\ \mathbf{l}.\mathbf{m} & \mathbf{l}.\mathbf{n} \end{vmatrix} = k^\alpha l^\beta \begin{vmatrix} m_\alpha & n_\alpha \\ m_\beta & n_\beta \end{vmatrix} = {}^1/_2\, \overline{kl}^{\alpha\beta}\, \overline{mn}_{\alpha\beta} = \mathbf{a}.\mathbf{b}$$

14,10. We have $\mathbf{a}.\mathbf{e}_{12} = {}^1/_2\, a_{\alpha\beta}\mathbf{e}^{\alpha\beta}.\mathbf{e}_{12}$. If α or β is 3 or 4 $\mathbf{e}^{\alpha\beta}$ contains a direction perpendicular to $\mathbf{e}_{12}$ and $\mathbf{e}^{\alpha\beta}.\mathbf{e}_{12}$ vanishes. Thus the only finite terms are those given by $\alpha = 1$ $\beta = 2$ and $\alpha = 2$ $\beta = 1$, and the right side of the equation is a_{12}.

We denote the projection of $\mathbf{a}$ on $\mathbf{e}_{12}$ by $\mathbf{b} = \kappa\mathbf{e}_{12}$. Then

$$a_{12} = \mathbf{a}.\mathbf{e}_{12} = \mathbf{b}.\mathbf{e}_{12} = \kappa\mathbf{e}_{12}.\mathbf{e}_{12} = \kappa g_{1212}$$

and the projection $\mathbf{b}$ is $a_{12}\mathbf{e}_{12}/g_{1212}$.

14,11. Let X denote the head of vector $\mathbf{x}$, and so for the other vectors. Let the two triangles that make the film be XTS XYT. The six area vectors are the halves of

$(\mathbf{x} - \mathbf{s})(\mathbf{t} - \mathbf{x})$ $(\mathbf{y} - \mathbf{x})(\mathbf{t} - \mathbf{x})$ $\mathbf{yx}$ $\mathbf{xs}$ $\mathbf{st}$ $\mathbf{ty}$ and when the first two are multiplied out the sum is seen to be zero.

14,12. In $\mathbf{a}\cdot\mathbf{b} = {}^1/_2\, a^{\theta\phi}b_{\theta\phi}$ we put $\mathbf{a} = \mathbf{e}_{\alpha\beta}$ $\mathbf{b} = \mathbf{e}_{\gamma\delta}$. Then $a^{\alpha\beta} = -a^{\beta\alpha} = 1$ and the other $a^{\theta\phi}$ vanish; $b^{\gamma\delta} = -b^{\delta\gamma} = 1$ and the other $b^{\theta\phi}$ vanish. Thus

$$\begin{aligned}\mathbf{e}_{\alpha\beta}\cdot\mathbf{e}_{\gamma\delta} &= b_{\alpha\beta} = g_{\alpha\theta}g_{\beta\phi}b^{\theta\phi} \\ &= g_{\alpha\gamma}g_{\beta\delta}b^{\gamma\delta} + g_{\alpha\delta}g_{\beta\gamma}b^{\delta\gamma} = g_{\alpha\beta\gamma\delta}\end{aligned}$$

Again we put $\mathbf{a} = \mathbf{e}_{12}$ $\mathbf{b} = \mathbf{e}^{\rho\sigma}$ so that $a^{12} = -a^{21} = 1$ and the other $a^{\theta\phi}$ vanish; $b_{\rho\sigma} = -b_{\sigma\rho} = 1$ and the other $b_{\theta\phi}$ vanish. Hence $\mathbf{e}_{12}\cdot\mathbf{e}^{\rho\sigma} = {}^1/_2\, a^{12}b_{12} + {}^1/_2\, a^{21}b_{21} = a^{12}b_{12} = b_{12}$.

And b_{12} is 1 if $\rho = 1 \quad \sigma = 2$, it is -1 if $\rho = 2 \quad \sigma = 1$, otherwise it is zero.

15,11. The vanishing of $\mathbf{v} \cdot \mathbf{w}$ is the condition that each volume vector should contain the direction perpendicular to the other.

We take $\mathbf{v}$ in the form $\mathbf{klm}$ and $\mathbf{w}$ in the form $\mathbf{rst}$. Let $\mathbf{p} = \kappa\mathbf{k} + \lambda\mathbf{l} + \mu\mathbf{m} \quad \mathbf{q} = \rho\mathbf{r} + \sigma\mathbf{s} + \tau\mathbf{t}$ so that $\mathbf{p}$ lies in $\mathbf{v}$ and $\mathbf{q}$ in $\mathbf{w}$. The condition that $\mathbf{q}$ is perpendicular to $\mathbf{v}$ is that $\mathbf{q} \cdot \mathbf{p}$ vanishes for all values of $\kappa \quad \lambda \quad \mu$. $\mathbf{q}$ will satisfy this condition and at the same time lie in $\mathbf{w}$ if we can determine $\rho \quad \sigma \quad \tau$ so that $(\rho\mathbf{r} + \sigma\mathbf{s} + \tau\mathbf{t}) \cdot \mathbf{k} \quad (\rho\mathbf{r} + \sigma\mathbf{s} + \tau\mathbf{t}) \cdot \mathbf{l} \quad (\rho\mathbf{r} + \sigma\mathbf{s} + \tau\mathbf{t}) \cdot \mathbf{m}$ vanish; the condition for which is

$$0 = \begin{vmatrix} \mathbf{k} \cdot \mathbf{r} & \mathbf{k} \cdot \mathbf{s} & \mathbf{k} \cdot \mathbf{t} \\ \mathbf{l} \cdot \mathbf{r} & \mathbf{l} \cdot \mathbf{s} & \mathbf{l} \cdot \mathbf{t} \\ \mathbf{m} \cdot \mathbf{r} & \mathbf{m} \cdot \mathbf{s} & \mathbf{m} \cdot \mathbf{t} \end{vmatrix} = k^{\alpha} l^{\beta} m^{\gamma} \begin{vmatrix} r_{\alpha} & s_{\alpha} & t_{\alpha} \\ r_{\beta} & s_{\beta} & t_{\beta} \\ r_{\gamma} & s_{\gamma} & t_{\gamma} \end{vmatrix}$$

$$= {}^{1}/_{6}\, \overline{klm}^{\alpha\beta\gamma} \overline{rst}_{\alpha\beta\gamma} = \mathbf{v} \cdot \mathbf{w}$$

15,12. We have $\mathbf{v} \cdot \mathbf{e}_{123} = {}^{1}/_{6} v_{\alpha\beta\gamma} \mathbf{e}^{\alpha\beta\gamma} \cdot \mathbf{e}_{123}$ and if α or β or γ is 4 $\mathbf{e}^{\alpha\beta\gamma}$ contains the normal to $\mathbf{e}_{123}$ so that the term vanishes. The only finite terms are the six for which $\alpha \quad \beta \quad \gamma$ are permutations of $1 \quad 2 \quad 3$, and the value of the right hand side is v_{123}.

We denote the projection of $\mathbf{v}$ on $\mathbf{e}_{123}$ by $\mathbf{w} = \kappa\mathbf{e}_{123}$. Then

$$v_{123} = \mathbf{v} \cdot \mathbf{e}_{123} = \mathbf{w} \cdot \mathbf{e}_{123} = \kappa \mathbf{e}_{123} \cdot \mathbf{e}_{123} = \kappa g_{123123}$$

so that the projection $\mathbf{w}$ is

$$v_{123} \mathbf{e}_{123} / g_{123123}$$

16,17. By article 7 $\quad \mathbf{e}_{\beta} \cdot \mathbf{c}^{\rho} = f_{\beta}^{\circ\rho}$

by article 2 $\quad d_{\alpha}^{\circ\beta} f_{\beta}^{\circ\rho} = \delta_{\alpha}^{\rho}$

and by article 6 $\quad \mathbf{e}_{\rho} = f_{\rho}^{\circ\sigma} \mathbf{c}_{\sigma}$

so that

$$\mathbf{C} \cdot \mathbf{E} = d_{\alpha}^{\circ\beta} \mathbf{e}^{\alpha} \mathbf{e}_{\beta} \cdot f_{\rho}^{\circ\sigma} \mathbf{c}^{\rho} \mathbf{c}_{\sigma} = \mathbf{e}^{\alpha} d_{\alpha}^{\circ\beta} f_{\beta}^{\circ\rho} \mathbf{e}_{\rho} = \mathbf{e}^{\alpha} \delta_{\alpha}^{\rho} \mathbf{e}_{\rho} = \mathbf{I}.$$

16,23. If $\mathbf{x}$ is converted to zero by forehand action

$$\varpi_{kx}\mathbf{p} + \varpi_{lx}\mathbf{q} + \varpi_{mx}\mathbf{r} + \varpi_{nx}\mathbf{s}$$

must vanish. By taking $\mathbf{x}$ normal to the triplane **klm** we can make the coefficients of $\mathbf{p}$ $\mathbf{q}$ $\mathbf{r}$ zero, but since $\mathbf{n}$ is not in **klm** the fourth term remains finite. By giving $\mathbf{x}$ an appropriate direction we can obtain zero components in any three directions, but a finite component will remain in a direction perpendicular to these.

The reasoning is similar for backhand action.

16,24. The forehand stroke of $\mathbf{kp} + \mathbf{lq} + \mathbf{mr}$ converts to zero a vector perpendicular to **klm**, and the backhand stroke one perpendicular to **pqr**; a single direction in each case.

The vectors $\mathbf{x}$ converted to zero by the forehand stroke of $\mathbf{kp} + \mathbf{lq}$ are given by $\mathbf{k} \bullet \mathbf{x} = \mathbf{l} \bullet \mathbf{x} = 0$, that is they lie in a biplane. And so for the backhand stroke.

All the vectors in the triplane $\mathbf{k} \bullet \mathbf{x} = 0$ are reduced to zero by the forehand stroke of **kp**, and all in $\mathbf{p} \bullet \mathbf{x} = 0$ by the backhand stroke.

17,3. The vector $\mathbf{e}^a$ converts to $\mathbf{c}^a$ since

$$\mathbf{R} \bullet \mathbf{e}^a = \mathbf{c}_\rho \mathbf{e}^\rho \bullet \mathbf{e}^a = g^{a\rho}\mathbf{c}_\rho = h^{a\rho}\mathbf{c}_\rho = \mathbf{c}^a$$

and the congruence follows from the fact that all size-numbers are left unaltered.

18,3. When $\mathbf{x}$ lies in $\mathbf{e}_{123}$ $\mathbf{e}^4 \bullet \mathbf{x}$ vanishes so that $\mathbf{Q}_{123} \bullet \mathbf{x} = \mathbf{x}$. Also $\mathbf{e}^4 \bullet \mathbf{e}_4 = 1$ so that $\mathbf{Q}_{123} \bullet \mathbf{e}_4 = \mathbf{e}_4 - 2\mathbf{e}^4/g^{44}$, and $\mathbf{e}^4 \bullet \mathbf{e}^4 = g^{44}$ so that $\mathbf{Q}_{123} \bullet \mathbf{e}^4 = \mathbf{e}^4 - 2\mathbf{e}^4 = -\mathbf{e}^4$.

20,4. We have to find what values of $d_4^{\cdot 4}$ will satisfy

$$g_{\rho\sigma} d_4^{\cdot\rho} d_4^{\cdot\sigma} = g_{44}$$

in which $d_4^{\cdot 1} = u^1\sqrt{g_{44}}$ $d_4^{\cdot 2} = u^2\sqrt{g_{44}}$ $d_4^{\cdot 3} = u^3\sqrt{g_{44}}$

and $\mathbf{u}$ is a velocity vector. We put

$$d_4^{\cdot a} = \sqrt{g_{44}}(u^a + z^a) \quad a = 1\ 2\ 3\ 4$$

The first equation then gives

$$g_{44} = g_{\rho\sigma}g_{44}(u^\rho + z^\rho)(u^\sigma + z^\sigma)$$

or

$$1 = g_{\rho\sigma}u^\rho u^\sigma + 2g_{\rho\sigma}u^\rho z^\sigma + g_{\rho\sigma}z^\rho z^\sigma$$

In accordance with the data we make $z^1 \quad z^2 \quad z^3$ vanish and the equation becomes

$$2z^4 g_{4\rho}u^\rho + z^4 z^4 g_{44} = 0$$

and has the two roots

$$z^4 = 0 \qquad z^4 = -2g_{4\rho}u^\rho/g_{44} = -2u_4/g_{44}$$

so that d_4^{o4} has the two values

$$u^4\sqrt{g_{44}} \qquad u^4\sqrt{g_{44}} - 2u_4/\sqrt{g_{44}}$$

Considered geometrically we have to find a vector **x** which has size-number g_{44} and has $x^1 = u^1\sqrt{g_{44}}$ $x^2 = u^2\sqrt{g_{44}}$ $x^3 = u^3\sqrt{g_{44}}$. This says that the head of the vector is a point of intersection of the contour of size-number g_{44} by the straight line given by the values of $x^1 \; x^2 \; x^3$. This straight line is parallel to $\mathbf{e}_4$ and therefore perpendicular to the triplane $\mathbf{e}^{123}$. The figure is thus symmetrical about the triplane (for the system of contour trisurfaces has complete symmetry with respect to any timelike vector at the origin and with respect to any spacelike triplane through the origin) and the two intersections are images of one another in it. The head of the vector $\mathbf{u}\sqrt{g_{44}}$ satisfies the conditions and is one of the intersections, so that the other is the head of its image, that is to say (article 18,2) the head of

$$(\mathbf{I} - 2\mathbf{e}_4\mathbf{e}_4/g_{44})\cdot\mathbf{u}\sqrt{g_{44}} \qquad \text{or} \qquad \mathbf{u}\sqrt{g_{44}} - 2u_4\mathbf{e}_4/\sqrt{g_{44}}$$

20,9. The classical velocity expressed as a vector is $\overrightarrow{ML}/\sigma_{OM}$. Since $\sigma_{OM} = \varpi_{\mathbf{kl}}/\sigma_{\mathbf{k}}$ we have $\overrightarrow{OM} = \mathbf{k}\varpi_{\mathbf{kl}}/\nu_{\mathbf{k}}$ and

$$\begin{aligned}\nu_{\mathbf{k}}\overrightarrow{ML} &= \nu_{\mathbf{k}}(\mathbf{l} - \overrightarrow{OM}) = \mathbf{l}\nu_{\mathbf{k}} - \mathbf{k}\varpi_{\mathbf{kl}}\\ &= k^\alpha k_\alpha l^\beta \mathbf{e}_\beta - l^\alpha k_\alpha k^\beta \mathbf{e}_\beta = \overline{kl}^{\alpha\beta}k_\alpha\mathbf{e}_\beta = {}^1/_2\,\overline{kl}^{\alpha\beta}\overline{k\mathbf{e}}_{\alpha\beta}\end{aligned}$$

and the required velocity is

$$(\mathbf{l}\nu_{\mathbf{k}} - \mathbf{k}\varpi_{\mathbf{kl}})/\sigma_{\mathbf{k}}\varpi_{\mathbf{kl}} \tag{1}$$

or

$$^1/_2\, \overline{kl}^{\alpha\beta}\overline{k\mathbf{e}}_{\alpha\beta}/\sigma_{\mathbf{k}}\varpi_{\mathbf{kl}} \tag{2}$$

We now consider the vector

$$\mathbf{q} \equiv {}^1/_2\, \overline{kl}^{\alpha\beta}\overline{k\mathbf{e}}_{\alpha\beta} \equiv {}^1/_2\, \overline{kl}_{\rho\sigma}\overline{k\mathbf{e}}^{\rho\sigma}$$

Since $\overline{k\mathbf{e}}_{\alpha\beta} \cdot \overline{k\mathbf{e}}^{\rho\sigma}$ written out in full is

$$(k_\alpha\mathbf{e}_\beta - k_\beta\mathbf{e}_\alpha)\cdot(k^\rho\mathbf{e}^\sigma - k^\sigma\mathbf{e}^\rho)$$

or

$$k_\alpha k^\rho \delta^\sigma_\beta - k_\alpha k^\sigma \delta^\rho_\beta - k_\beta k^\rho \delta^\sigma + k_\beta k^\sigma \delta^\rho_\alpha$$

we can express $\mathbf{q}\cdot\mathbf{q}$ as the sum of four quantities of which one is

$$\phi = {}^1/_4\, \overline{kl}^{\alpha\beta}\overline{kl}_{\rho\sigma}k_\alpha k^\rho \delta^\sigma_\beta = {}^1/_4\, \overline{kl}^{\alpha\beta}k^\rho \overline{kl}_{\rho\beta}k_\alpha$$

By interchanging α β and adding the two expressions we have

$$2\phi = {}^1/_4\, \overline{kl}^{\alpha\beta}k^\rho(\overline{kl}_{\rho\beta}k_\alpha - \overline{kl}_{\rho\alpha}k_\beta)$$

or since

$$\overline{kl}_{\alpha\beta}k_\rho + \overline{kl}_{\beta\rho}k_\alpha + \overline{kl}_{\rho\alpha}k_\beta = \overline{klk}_{\alpha\beta\rho} = 0$$

$$2\phi = {}^1/_4\, \overline{kl}^{\alpha\beta}k^\rho \overline{kl}_{\alpha\beta}k_\rho$$

so that by article 12,6

$$4\phi = \nu_{\mathbf{k}}\nu_{\mathbf{kl}}$$

Each of the other three quantities has the same value as ϕ, so that the size number of $\mathbf{q}$ is $\nu_{\mathbf{k}}\nu_{\mathbf{kl}}$ and the size-number of expression (2) is

$$\nu_{\mathbf{k}}\nu_{\mathbf{kl}}/\sigma^2_{\mathbf{k}}\varpi^2_{\mathbf{kl}} \qquad \text{or} \qquad \nu_{\mathbf{kl}}/\varpi^2_{\mathbf{kl}}$$

23,13. When the three vectors lie in the same biplane the angle ω is either 0 or π. In the former case we have

$$\gamma^2(1 - \alpha\beta)^2 = (1 - \alpha\beta)^2 - (1 - \alpha^2)(1 - \beta^2) = (\alpha - \beta)^2$$

and the two values of γ are

$$(\alpha - \beta)/(1 - \alpha\beta) \quad \text{and} \quad (\beta - \alpha)/(1 - \alpha\beta)$$

When $\omega = \pi$

$$\gamma^2(1 + \alpha\beta)^2 = (1 + \alpha\beta)^2 - (1 - \alpha^2)(1 - \beta^2) = (\alpha + \beta)^2$$

so that $\gamma = (\alpha + \beta)/(1 + \alpha\beta)$. We ignore the negative value as that only means measuring the velocity in the reverse direction.

23,14. To the second order of small quantities

$$\begin{aligned} 1 - \gamma^2 &= (1 - \alpha^2)(1 - \beta^2)(1 + 2\alpha\beta \cos \omega) \\ &= 1 - \alpha^2 - \beta^2 + 2\alpha\beta \cos \omega \end{aligned}$$

24,7. In seconds the radius is

$$6{\cdot}371 \times 10^8 \div 3 \times 10^{10} = 2{\cdot}124 \times 10^{-2}$$

A point on the equator travels a distance of $2\pi \times 2{\cdot}124 \times 10^{-2}$ seconds in a time-lapse of 86,400 seconds, so that

$$v = 2\pi \times 2{\cdot}124 \times 10^{-2} \div 86{,}400 = 1{\cdot}544 \times 10^{-6}$$

Let a pair of triplanes perpendicular to E's world line and close together cut elements $d\mathbf{k}$ $d\mathbf{l}$ from the world lines of E and S. Then the relation $\sigma_{d\mathbf{l}} = \sigma_{d\mathbf{k}}\sqrt{1 - v^2}$ is true of the elements and therefore of finite portions of the lines. E's day being 86,400 seconds, that of S is $86{,}400\ (1 - v^2/2)$ seconds, and the difference is (in seconds)

$$86{,}400 \times 1{\cdot}544^2 \times 10^{-12} \div 2$$
$$\text{or} \qquad 10^{-7}.$$

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$\mathbf{e}_\alpha$ $\mathbf{e}^\alpha$ prime vector 2,2 7,1

$\mathbf{e}_{\alpha\beta}$ $\mathbf{e}^{\alpha\beta}$ prime area-vector 2,4

$\mathbf{e}_{\alpha\beta\gamma}$ $\mathbf{e}^{\alpha\beta\gamma}$ prime volume-vector 2,5

$\overrightarrow{OP}$ $\mathbf{l}$ $\mathbf{m}$ line-vector 1,3 2,2

$\mathbf{a}$ $\mathbf{b}$ $\mathbf{lm}$ area-vector 1,3

$\mathbf{v}$ $\mathbf{w}$ $\mathbf{lmn}$ volume-vector 1,11

f $\mathbf{lmnp}$ fourfold volume or supervolume 1,17

l^α l_α line-vector components 2,2 4,7

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$v^{\alpha\beta\gamma}$ $v_{\alpha\beta\gamma}$ $\overline{lmn}^{\alpha\beta\gamma}$ $\overline{lmn}_{\alpha\beta\gamma}$ volume-vector components 12,7

ν_{OP} $\nu_{\mathbf{l}}$ $\mathbf{l}\cdot\mathbf{l}$ size-number of line-vector 3,3 4,5

$\nu_{\mathbf{a}}$ $\nu_{\mathbf{lm}}$ $\mathbf{a}\cdot\mathbf{a}$ size-number of area-vector 5,4 5,7 12,6

$\nu_{\mathbf{v}}$ $\nu_{\mathbf{lmn}}$ $\mathbf{v}\cdot\mathbf{v}$ size-number of volume-vector 6,5 6,8 12,8

ν_f $\nu_{\mathbf{lmnp}}$ size-number of supervolume 7,3

σ_{OP} $\sigma_{\mathbf{l}}$ size of line-vector 3 3

$\sigma_{\mathbf{a}}$ $\sigma_{\mathbf{lm}}$ size of area-vector 5,4

$\sigma_{\mathbf{v}}$ $\sigma_{\mathbf{lmn}}$ size of volume-vector 6,5

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$\mathbf{lm}$ $\mathbf{lmn}$ $\mathbf{lmnp}$ outer products of line-vectors 12,1 ?

$\varpi_{\mathbf{lm}}$ $\mathbf{l}\cdot\mathbf{m}$ inner product of line-vectors 4,4 4,5

$\varpi_{\mathbf{ab}}$ $\varpi_{\mathbf{lm},\mathbf{rs}}$ $\mathbf{a}\cdot\mathbf{b}$ inner product of area-vectors 14,7 14,8

$\varpi_{\mathbf{vw}}$ $\varpi_{\mathbf{lmn},\mathbf{rst}}$ $\mathbf{v}\cdot\mathbf{w}$ inner product of volume-vectors 15,7 15,8

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$\mathbf{A}\cdot\mathbf{B}$ $\mathbf{AB}$ product of operators 13,4

$\mathbf{x}\cdot\mathbf{A}$ $\mathbf{A}\cdot\mathbf{x}$ ${}_{\mathbf{x}}\mathbf{A}$ $\mathbf{A}_{\mathbf{x}}$ $\mathbf{x}\cdot\mathbf{A}\cdot\mathbf{y}$ ${}_{\mathbf{x}}\mathbf{A}_{\mathbf{y}}$ action of operator 13,2 13,6

$\bar{\mathbf{A}}$ conjugate operator 13,4

$\mathbf{I}$ identity operator 13,11

$\equiv$ 1,5

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$\sum_{\alpha\beta\gamma\delta}$ 20,11

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